An Implied Default Dependency Model of a Credit Portfolio based on the Number of Defaults

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Abstract

We propose a market-implied default dependency model that can be exactly calibrated to market quotes of credit index tranches and single name credit default swaps belonging to their reference portfolio. Its calibration does not need simulation even in heterogeneous setting, hence is robust and quite rapidly. The Gaussian copula model is the market standard for default dependency, though it appears to have a problem called *correlation smile* that the model parameter changes for each tranche belonging to a credit index. In addition, without the assumption of a large homogeneous portfolio, its calibration generally needs simulation; however, under this assumption we cannot evaluate the risk of tranches with respect to each company. On the other hand, in light of the market crash of 2008, there is a strong demand for default dependency models that have good tractability and high accuracy. The implied factor-copula and top-down approaches are currently applied for research of this field. A model that satisfies the practical demands in a heterogeneous setting has not yet been reported. In this paper, we present a representation of joint default probability of a credit portfolio by a minimal entropy martingale measure. We derive the above model under a feasible assumption related to the market. We calibrate our model to market quotes of CDX NA IG tranches and evaluate their risks.

1 Introduction

In the focus of this paper is on the market of single name credit default swaps (CDSs), credit indices, and their tranches. CDS is a basic tool for credit risk transfer. It is a bilateral derivative contract between a seller of protection and a buyer of protection. The protection seller covers the default loss of the reference asset up to the maturity date of the contract. For this, the protection buyer pays a premium to the seller up to the maturity date if there is no credit event. If there is a credit event, the trade is settled and the premium stops being paid.

Credit indices are equally weighted portfolio of CDSs. They have higher liquidity than single-name CDSs. Companies in a portfolio are selected by region, credit grade, industrial sector and market activity. Credit indices are benchmarks of the market view on the credit risk of their relevant economies. CDX and iTraxx are the main names. Credit index tranches allow investors to gain exposure between their

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attachment and detachment points of the index loss distribution, the same as for Collateralized Debt Obligations. For details of CDX/iTraxx, see *Markit Credit Indices Primer*.

The big difference between a credit index and its tranches is that the price and risk of tranches depend on default dependency among the companies in their reference portfolio. In most cases, senior tranches are safer than junior ones since they are protected by their seniority. But in an extreme case such as when all defaults are perfectly correlated, seniority has no meaning. Credit index tranches are regarded as the benchmark of default dependency.

The Gaussian copula model introduced by Li [2001] is the market standard for default dependency. However, it is well known that the model is not fully consistent with the market view. Because of this incompatibility, the model parameter changes for each tranche. This problem is called *correlation smile*. There is also another issue; that calibration of the Gaussian copula model needs simulation without a large homogeneous pool (LHP) approximation. It assumes that a portfolio is comprised of identical assets, and that each pair of reference credits in the portfolio has the same correlation. LHP approximation is not significant problem in pricing, but the risks of each tranche can not be evaluated with respect to each underlying CDS under this approximation.

The implied-factor-copula and top-down approaches are the ones currently applied in research in this field. The aim of implied-factor-copula approach is to reduce the phenomenon of the "correlation smile" by improvement of the copula models. Copula models can be formulated as a conditionally independent model with a latent factor. Nedeljkovic et.al [2009] used a nonparametric latent factor, and Hull and White [2006, 2009] generalized a set of hazard rates corresponding to each latent factor value. Burtschell et al. [2009] gave comprehensive analyses of the factor-copula models. Andersen et.al [2003] proposed a factor-copula model to speed up of computations, but not to reduce the smile.

The top-down approach is based on the fact that for most of a credit portfolio's derivatives, the cash flow depends only on the aggregate loss process, not on the names of defaults. It directly models the aggregate loss process. If they are needed, single name default processes are derived to be consistent with the aggregate loss by *random thinning operation*. We refer here, from among various relevant papers, to Bielecki et al. [2008] and earlier research by Gieseke et al. [2009]. Schönbucher and Ehlers [2006] discussed canonical construction (constructing models from hazard rates) of the top-down approach.

Assets in the credit market can be classified into two categories. Let us call credit assets whose prices are function of (joint) default probability only at the evaluation date as static, and the rest as dynamic. As is well known, CDSs, credit indices and their tranches are all static. For example, options of corporate bonds and CDSwaptions (options on forward starting CDSs) are dynamic. To price dynamic credit assets, we have to specify the *dynamics* of (joint) default probability. However, to price static credit assets, we can concentrate on finding joint default probability on a pricing measure at an evaluation date, and the dynamics of the default probability are not an issue. The development of interest rate models suggests to us that in the evaluation of static assets, it is best to exclude any specification of dynamics other than the defaults of companies.

Our goal is to find an implied joint default probability in the market of CDSs, credit indices, and their tranches with neither the specification of the dynamics nor homogeneous assumption. For this purpose, a top-down approach is not suitable. Since it is essentially parametric and dynamic, and its best advantage is its homogeneous setting, an implied-factor-copula approach can be one of the solutions. However, we adopt another way since we seek more tractability and less arbitrariness.

Although we only consider joint default probability at the evaluation date, the market is still incomplete. There exist several ways to choose a pricing measure in incomplete markets; for example, minimal martingale measure (MMM), variance-optimal martingale measure (VMM), utility martingale measure (UMM), or minimal entropy martingale measure (MEMM). For our purpose, the most important factor is tractability. Among these, we consider that MEMM has the best tractability for our setup. MEMM introduced by Miyahara [1996, 1999] minimizes relative entropy between the pricing and the physical measures. It is known that MEMM is related to the classical utility maximization approach with exponential utility. The concept of entropy was originally developed in thermodynamics; though its applications are found in physics, chemistry, information science, and finance. Results of relative-entropy minimization often agree with intuitive guesses, since entropy is built into natural laws. Shouda [2001] proposed a martingale measure of single name CDSs that has minimal entropy with respect to a given bilateral default dependency, and evaluated n-th to default swaps on it.

The rest of this paper is structured as follows. In the second section, we set up the market and derive conditions whose pricing measures must hold. We show that MEMM is represented by risk premium for the number of defaults and the default of individual companies. In the third section, we make an additional assumption for the market and derive a tractable model of implied joint default probability as a MEMM in a subset of pricing measures. Market practice is given in the fourth section. We also calibrate our model with respect to the market quotes of single name CDSs and CDX NA IG tranches, and evaluate the risks of tranches. The fifth section concludes this report.

2 GENERAL FRAMEWORK

2.1 The Model of a Defaultable Portfolio

Consider a credit portfolio constructed by N of companies. Let the default time of the company $k = 1, \ldots, N$ be denoted by $\tau_k : \Omega \to R^+$ on a probability space $(\Omega, \mathcal{F}, P)$. Let P be the physical measure and denote an entire set of P-equivalent measures by $\mathcal{Q}$. Define the default indicators $D_t^k := 1_{\tau_k \leq t}$ as jump processes associated with $\tau_.$ and their natural filtration[1]

$$\mathcal{F}_t^k := \sigma(\{D_s^k\} | s \leq t), \quad \mathcal{F}_t := \bigvee_{k=1}^{N} \mathcal{F}_t^k \tag{1}$$

τ_k is $(\mathcal{F}_t^k)$-stopping time. Denote the number of defaults up to time t as

$$N_t := \sum_{k=1}^{N} D_t^k \tag{2}$$

and the filtration generated by N_t as

$$\mathcal{G}_t := \sigma(N_s | s \leq t) \tag{3}$$

Evidently, $\mathcal{G}_t \subseteq \mathcal{F}_t$. In this article, we do not specify the market filtration $(\mathcal{F}_t^M)_{t \geq 0}$ that represents information revealed to the market up to time t and on which asset prices at time t are evaluated. We only assume that it equals the natural filtration of defaults at the evaluation date, i.e., $\mathcal{F}_0^M = \mathcal{F}_0$ and the defaults of companies are $\mathcal{F}_t^M$-measurable, i.e. $\mathcal{F}_t^M \supseteq \mathcal{F}_t$ for $t \geq 0$.[2]

2.2 The Market

Suppose a market in which there exist single name CDSs and credit index tranches written on a credit portfolio. Assume that the market is arbitrage free. Thus there exists at least one P-equivalent martingale

[1] $\mathcal{F}_t$ is not right continuous (see e.g. Protter [2004], p. 370). In this paper, this property is not important.

[2] See Bielecki and Rutkowski [2002] pp. 221-245 for the valuation of defaultable claims on enlarged market filtrations.

measure $Q \in \mathcal{Q}$ on which any price of a trading asset equals to the expectation of the discounted cash flows generated by the asset. The default free discount function up to time t is given by B_t, and assume that it is deterministic.

Discretizing the time axis by $\{T_i\}_{i=0,\ldots,M}$, $T_0 = 0$, $T_{i-1} < T_i$. Assume that all instruments under our consideration have no cash flow paid out of $\{T_i\}$. The present value of a single name CDS premium leg and a protection leg written on the company k, maturating at T_j are described by a linear equation of $Q(D^k_{T_i} = 1)$, $i = 0, \ldots, j$ respectively as;

$$p^{k,j}_{\text{premium}} = E^Q\Big[\sum_{i=1}^{j} c^k_i B_{T_i}(1 - D^k_{T_i})\Big] = \sum_{i=1}^{j} c^k_i B_{T_i} Q(D^k_{T_i} = 0) \tag{4}$$

$$p^{k,j}_{\text{protection}} = E^Q\Big[\sum_{i=1}^{j} L^k_i B_{T_i}(D^k_{t_{i-1}} - D^k_{T_i})\Big] = \sum_{i=1}^{j} L^k_i B_{T_i}(Q(D^k_{t_{i-1}} = 1) - Q(D^k_{T_i} = 1)) \tag{5}$$

where c^k_i denotes the running spread for the period $[T_{i-1}, T_i]$ and L^k_i denotes the expected loss given default of company k associated with the payment at T_i. Assume that $L^\cdot_\cdot$ are deterministic. We do not specify the CDSs in the market, but instead, simply assume that we know $Q(D^k_{T_i} = 1)$ for all $i = 1, \ldots, M$, $k = 1, \ldots, N$ from the market quotes of single name CDS as

$$\text{(M1)} \qquad Q(D^k_{T_i} = 1) = q^k_i, \quad (i = 1, \ldots, M, k = 1, \ldots, N)$$

Next, consider a credit index tranche written on the credit portfolio with an attachment point x, and a detachment point 1. The cash flow c_i will be paid to its premium leg holder at time T_i proportional to the remaining principal amount at time T_i. Instead, the loss of principal amount from T_{i-1} to T_i will be paid to its protection leg holder at time T_i. Denote the expected loss given default associated with the payment at T_i by L_i. Assume that $L_\cdot$ are deterministic.

The remaining principal amount of the tranche at time t is given by $1 - (N_t - xN)^+/(1-x)N$, thus the present value of premium leg equals to;

$$\begin{aligned} p_{\text{premium}}(x) &= E^Q\Big[\sum_{i=1}^{M} c_i B_{T_i}\left(1 - \frac{(N_{T_i} - xN)^+}{(1-x)N}\right)\Big] \\ &= \sum_{i=1}^{M} c_i B_{T_i} \sum_{n=0}^{N}\left(1 - \frac{(n - xN)^+}{(1-x)N}\right) Q(N_{T_i} = n) \end{aligned} \tag{6}$$

On the other hand, the present value of protection leg equals to;

$$\begin{aligned} p_{\text{protection}}(x) &= E^Q\Big[\sum_{i=1}^{M} L_i B_{T_i} \frac{(N_{T_i} - xN)^+ - (N_{T_{i-1}} - xN)^+}{(1-x)N}\Big] \\ &= \sum_{i=0}^{M}\sum_{m=0}^{N-1} Q(N_{T_i} = m)\Big(1_{i<M} L_{i+1} B_{T_{i+1}} - 1_{i>0} L_i B_{T_i}\Big) \sum_{n=m+1}^{N} \frac{(n - xN)^+ - (n - 1 - xN)^+}{(1-x)N} \end{aligned} \tag{7}$$

(6) and (7) say that present value of credit index tranches are a linear function of the distribution of $N_{T_i}, i = 0, \ldots, M$. Assume that there exist L of tranches written on the defaultable portfolio. From the present value of tranches we get the following L of equations;

$$\text{(M2)} \qquad \sum_{i=0}^{M}\sum_{n=0}^{N} A^{in}_l Q(N_{T_i} = n) = b_l, \quad (l = 1, \ldots, L)$$

The explicit forms of $A^{\cdot\cdot}_\cdot$ and $b_\cdot$ are given in the appendix.

2.3 Minimal Entropy Martingale Measure

On any P-equivalent measure that holds (M1) and (M2), we get the market price of CDSs and credit index tranches as the expectation of the discounted cash flows generated by them. So we define

Definition 1 *We call P-equivalent measure Q a* **pricing measure** *if it holds (M1) and (M2).*

We denote the entire set of P-equivalent measures on those (M1) and (M2) held by $\mathcal{Q}_M^1 \subseteq \mathcal{Q}$ and $\mathcal{Q}_M^2 \subseteq \mathcal{Q}$, respectively. The entire set of pricing measures is represented by $\mathcal{Q}_M = \mathcal{Q}_M^1 \cap \mathcal{Q}_M^2$. We also define a class of measures

Definition 2 *We call P-equivalent measure Q_I an* **independent-default measure** *if the default time of companies (τ_k) are mutually independent on Q_I, i.e.,*

$$Q_I\Big(\bigcap_{k=1}^{N}\{\tau_k \le t_k\}\Big) = \prod_{k=1}^{N} Q_I(\tau_k \le t_k) \tag{8}$$

for any $(t_k) \in [0, T_M]^N$.

Denote the entire set of independent-default measures by $\mathcal{Q}_I$. We assume that there exists an independent-default measure Q_0 that holds (M1), i.e., $Q_0 \in \mathcal{Q}_I \cap \mathcal{Q}_M^1$. We call this **marginal pricing measure**. Note that $Q_0 \notin \mathcal{Q}_M^2$ in general, and we do not assume its uniqueness.

We define *a fitness criterion*

$$G(Q) := E^Q\big[\ln \rho(Q)\big], \quad \rho(Q) := \frac{dQ}{dQ_0}, \quad (Q \in \mathcal{Q}) \tag{9}$$

$\rho(Q)$ represents the Radon-Nikodym derivative of Q with respect to Q_0.

A pricing measure that minimizes $G(Q)$ is known as the minimal entropy martingale measure (MEMM). If the marginal distributions of defaults on Q are equal to those on Q_0, $G(Q)$ is called mutual information in information science.

Define sigma algebras $\tilde{\mathcal{G}}$, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}_k$, $k = 1, \ldots, N$ those represent discretized information,

$$\tilde{\mathcal{G}} := \bigvee_{i=0}^{M} \sigma\big(N_{T_i}\big), \quad \tilde{\mathcal{F}}_k := \bigvee_{i=0}^{M} \sigma\big(D_{T_i}^k\big), \quad \tilde{\mathcal{F}} := \bigvee_{k=1}^{N} \tilde{\mathcal{F}}_k \tag{10}$$

Evidently $\tilde{\mathcal{G}} \subseteq \mathcal{G}_{T_M}$, $\tilde{\mathcal{F}} \subseteq \mathcal{F}_{T_M}$, $\tilde{\mathcal{F}}_k \subseteq \mathcal{F}_{T_M}^k$, and $\tilde{\mathcal{G}} \subseteq \tilde{\mathcal{F}}$,

Proposition 3 *For any sigma algebra $\hat{\mathcal{F}}$ that holds $\mathcal{F} \supseteq \hat{\mathcal{F}} \supseteq \tilde{\mathcal{F}}$, and any pricing measure $Q \in \mathcal{Q}_M$, define a P-equivalent measure $\tilde{Q}$ such that*

$$\rho(\tilde{Q}) = E^{Q_0}\big[\rho(Q)\big|\hat{\mathcal{F}}\big] \tag{11}$$

Then $\tilde{Q}$ is also a pricing measure, i.e. $\tilde{Q} \in \mathcal{Q}_M$ and $G(\tilde{Q}) \le G(Q)$.

Proof.

$$\begin{aligned} Q(N_{T_i} = n) &= E^{Q_0}\big[\rho(Q) 1_{N_{T_i}=n}\big] = E^{Q_0}\Big[E^{Q_0}\big[\rho(Q) 1_{N_{T_i}=n}\big|\hat{\mathcal{F}}\big]\Big] \\ &= E^{Q_0}\Big[E^{Q_0}\big[\rho(Q)\big|\hat{\mathcal{F}}\big] 1_{N_{T_i}=n}\Big] = E^{Q_0}\big[\rho(\tilde{Q}) 1_{N_{T_i}=n}\big] = \tilde{Q}(N_{T_i} = n) \end{aligned} \tag{12}$$

for $i = 1, \dots, M$, $n = 0, \dots, N$. In the same way,

$$Q(D^k_{T_i} = 1) = \tilde{Q}(D^k_{T_i} = 1) \tag{13}$$

for $i = 1, \dots, M$, $k = 1, \dots, N$. Hence if Q holds (M1) and (M2), $\tilde{Q}$ also does.

Since $\rho(\tilde{Q})$ is $\hat{\mathcal{F}}$-measurable,

$$\begin{aligned} G(\tilde{Q}) &= E^{\tilde{Q}}\left[\ln \rho(\tilde{Q})\right] = E^{Q_0}\left[\rho(\tilde{Q}) \ln \rho(\tilde{Q})\right] = E^{Q_0}\left[E^{Q_0}\left[\rho(Q)\middle|\hat{\mathcal{F}}\right] \ln \rho(\tilde{Q})\right] \\ &= E^{Q_0}\left[\rho(Q) \ln \rho(\tilde{Q})\right] = E^{Q}\left[\ln \rho(\tilde{Q})\right] \end{aligned} \tag{14}$$

Thus from lemma 1 of Miyahara [1999]

$$G(Q) - G(\tilde{Q}) = E^Q\left[\ln \frac{dQ}{d\tilde{Q}}\right] \geq 0 \tag{15}$$

Q.E.D.

Definition 4 *For any sigma algebra $\hat{\mathcal{F}} \subseteq \mathcal{F}$ and P-equivalent measures Q and $\tilde{Q}$, we denote Q is* **$\hat{\mathcal{F}}$-equivalent** *to $\tilde{Q}$ or* **$(\tilde{Q}, \hat{\mathcal{F}})$-equivalent***, iff the Radon-Nikodym derivative of Q with respect to $\tilde{Q}$ is $\hat{\mathcal{F}}$-measurable.*

Denote the entire set of $(\tilde{Q}, \hat{\mathcal{F}})$-equivalent measures by $\tilde{\mathcal{Q}}(\tilde{Q}, \hat{\mathcal{F}})$.

$$\tilde{\mathcal{Q}}(\tilde{Q}, \hat{\mathcal{F}}) := \left\{ Q \in \mathcal{Q} \middle| \frac{dQ}{d\tilde{Q}} \text{ is } \hat{\mathcal{F}}\text{-measurable} \right\} \tag{16}$$

Then we can say that $\tilde{Q}$ in proposition 3 is $(Q_0, \hat{\mathcal{F}})$-equivalent. Proposition 3 says that, if there exists $\hat{Q}$ that minimizes $G(Q)$ among the $(Q_0, \tilde{\mathcal{F}})$-equivalent pricing measures, $\tilde{\mathcal{Q}}(Q_0, \tilde{\mathcal{F}}) \cap \mathcal{Q}_M$, $\hat{Q}$ also minimizes $G(Q)$ among the entire pricing measures $\mathcal{Q}_M$. The representation of such $\hat{Q}$ is given by the following theorem.

Theorem 5 (Representation of MEMM) *A pricing measure $\hat{Q}$ minimizes $G(Q)$ among the pricing measures $\mathcal{Q}_M$ iff $\rho(\hat{Q})$ is represented by a form*

$$\rho(\hat{Q}) = \frac{\exp\left(\sum_{l=1}^{L} \sum_{i=1}^{M} \sum_{n=0}^{N} \lambda_l A_l^{in} 1(N_{T_i} = n) + \sum_{i=1}^{M} \sum_{k=1}^{N} \zeta_i^k 1(D^k_{T_i} = 1)\right)}{E^{Q_0}\left[\exp\left(\sum_{l=1}^{L} \sum_{i=1}^{M} \sum_{n=0}^{N} \lambda_l A_l^{in} 1(N_{T_i} = n) + \sum_{i=1}^{M} \sum_{k=1}^{N} \zeta_i^k 1(D^k_{T_i} = 1)\right)\right]} \tag{17}$$

where parameters $\lambda_l, l = 1, \dots, L$ and $\zeta_i^k, i = 1, \dots, M, k = 1, \dots, N$ are determined to hold (M1) and (M2). If $\mathcal{Q}_M \neq \emptyset$, there exists such $\hat{Q}$ uniquely.

The parameters of (17) can be interpreted as being that λ_l represents the risk premium corresponding to the number of defaults at the range of l-th tranche, and ζ_i^k represents the risk premium corresponding to the default of k-th company during the period from T_{i-1} to T_i.

We can say that $\hat{Q}$ in theorem 5 is $\tilde{\mathcal{G}}$-equivalent to an independent-default measure. This property is worthy of note since independent-default measures are easy for understanding and for calculations. We can easily evaluate (M1) or (M2) on $\hat{Q}$ under the condition of $\tilde{\mathcal{G}}$. Although even for an independent-default measure Q_I, the calculation of probabilities corresponding to path of $N_{T.}$ is time consuming, since N_t is not $(Q_I, \mathcal{G}_t)$-Markovian in general. So we consider making a more tractable model in the next section. Incidentally, a top-down approach often assumes the existence of a pricing measure on which N_t becomes Markovian, to make tractable examples (Giesecke et al. [2009]). Meanwhile, we attempt another way.

3 A TRACTABLE MODEL

Suppose that all tranches have the same maturity at T_M, then their largest component of risk exposure is at T_M. So we put the following assumption into the market.

Assumption 6 *We assume that there exists a pricing measure that is $\tilde{\mathcal{G}}_M$-equivalent to an independent-default measure, i.e. $\tilde{\mathcal{Q}}_I \cap \mathcal{Q}_M \neq \emptyset$, where $\tilde{\mathcal{G}}_M := \sigma(N_{T_M})$ and*

$$\tilde{\mathcal{Q}}_I := \left\{ Q \in \mathcal{Q} \middle| \exists Q_I \in \mathcal{Q}_I \text{ s.t. } \frac{dQ}{dQ_I} \text{ is } \tilde{\mathcal{G}}_M\text{-measurable} \right\} = \bigcup_{Q_I \in \mathcal{Q}_I} \tilde{\mathcal{Q}}(Q_I, \tilde{\mathcal{G}}_M) \tag{18}$$

Consider (M1) (M2) and $G(\cdot)$ in $\tilde{\mathcal{Q}}_I$. For any independent-default measure $Q_I \in \mathcal{Q}_I$ and $(Q_I, \tilde{\mathcal{G}}_M)$-equivalent measure $\tilde{Q} \in \tilde{\mathcal{Q}}(Q_I, \tilde{\mathcal{G}}_M)$, we can define $\varrho_n : \tilde{\mathcal{Q}}_I \times \mathcal{Q}_I \to R^+$, for $n = 0, \ldots, N$ as to hold

$$\frac{d\tilde{Q}}{dQ_I} = \sum_{n=0}^{N} \varrho_n(\tilde{Q}|Q_I) 1_{N_{T_M}=n} \tag{19}$$

We often omit to write the argument of $\rho_n(\cdot|\cdot)$ if they are trivial in the context. To hold the completeness condition,

$$\sum_{n=0}^{N} \varrho_n Q_I(N_{T_M} = n) = 1 \tag{20}$$

$\tilde{Q}$, Q_I and $\varrho_\cdot$ hold the following equations

$$\tilde{Q}(D^k_{T_i} = 1) = \sum_{n=0}^{N} \varrho_n Q_I(D^k_{T_i} = 1, N_{T_M} = n) \tag{21}$$

$$\tilde{Q}(N_{T_i} = n) = \sum_{m=0}^{N} \varrho_m Q_I(N_{T_i} = n, N_{T_M} = m) \tag{22}$$

From (22), (M2) becomes

$$\text{(M2')} \qquad \sum_{i=0}^{M} \sum_{n=0}^{N} \sum_{m=0}^{N} \varrho_m A^{in}_l Q_I(N_{T_i} = n, N_{T_M} = m) = b_l \quad (l = 1, \ldots, L)$$

For (M1), we give the following lemma.

Lemma 7 *Suppose an independent-default measure $Q_I \in \mathcal{Q}_I$ and a $(Q_I, \tilde{\mathcal{G}}_M)$-equivalent measure $\tilde{Q} \in \tilde{\mathcal{Q}}(Q_I, \tilde{\mathcal{G}}_M)$. If $\tilde{Q}$ holds (M1), i.e. $\tilde{Q} \in \mathcal{Q}^1_M$, Q_I holds the following equation*

$$\frac{Q_I(D^k_{T_i} = 1)}{Q_I(D^k_{T_M} = 1)} = \frac{q^k_i}{q^k_M} \tag{23}$$

On the other hand, under the condition that Q_I holds (23), $\tilde{Q}$ holds (M1), i.e., $\tilde{Q} \in \mathcal{Q}^1_M$ if it holds (M1) only at $i = M$.

Proof. On any independent-default measure, N_T and D^k_t are mutually independent under the condition of D^k_T.

$$Q_I(N_T = n, D^k_t = 1|D^k_T) = Q_I(D^k_t = 1|D^k_T)Q_I(N_T = n|D^k_T) \tag{24}$$

So that since $Q_I(D^k_{T_i} = 1, D^k_{T_M} = 0) = 0$,

$$Q_I(N_{T_M} = n, D^k_{T_i} = 1) = Q_I(D^k_{T_i} = 1)Q_I(N_{T_M} = n|D^k_{T_M} = 1) \tag{25}$$

From (21),

$$\begin{aligned}\tilde{Q}(D^k_{T_i} = 1) &= \sum_{n=0}^{N} \varrho_n Q_I(D^k_{T_i} = 1, N_{T_M} = n) \\ &= Q_I(D^k_{T_i} = 1)\sum_{n=0}^{N} \varrho_n Q_I(N_{T_M} = n|D^k_{T_M} = 1) = Q_I(D^k_{T_i} = 1)\frac{\tilde{Q}(D^k_{T_M} = 1)}{Q_I(D^k_{T_M} = 1)} \end{aligned}\tag{26}$$

Q.E.D.

Suppose an independent-default measure $Q_I \in \mathcal{Q}_I$ that holds (23). Denote $h_k = Q_I(D^k_{T_M} = 1), k = 1, \ldots, N$. Then the following values become function of $h.$.

$$X_n(h.) := Q_I(N_{T_M} = n) \tag{27}$$
$$Y_{kn}(h.) := Q_I(D^k_{T_M} = 1|N_{T_M} = n) \tag{28}$$
$$V_i^{nm}(h.) := Q_I(N_{T_i} = m|N_{T_M} = n) \tag{29}$$

And if $Q_I \in \mathcal{Q}(Q_0, \tilde{\mathcal{F}})$, for any $(Q_I, \tilde{\mathcal{G}}_M)$-equivalent measure $\tilde{Q} \in \mathcal{Q}(Q_I, \tilde{\mathcal{G}}_M)$, the fitness criterion $G(\tilde{Q})$ becomes

$$G(\tilde{Q}) = E^{\tilde{Q}}\Big[\ln \frac{d\tilde{Q}}{dQ_0}\Big] = E^{\tilde{Q}}\Big[\ln \frac{d\tilde{Q}}{dQ_I} + \ln \frac{dQ_I}{dQ_0}\Big] = \sum_{n=0}^{N} X_n(h.)\varrho_n(\tilde{Q}|Q_I)\Big(\ln \varrho_n(\tilde{Q}|Q_I) + \Lambda_n(h.)\Big) \tag{30}$$

where

$$\Lambda_n(h.) := \sum_{k=1}^{N}\Big(Y_{kn}(h.)\ln \frac{h_k}{q^k_M} + (1 - Y_{kn}(h.))\ln \frac{1-h_k}{1-q^k_M}\Big) \tag{31}$$

To calculate $X_n(h.)$, $Y_{kn}(h.)$ and $V_i^{nm}(h.)$ from $h.$, we can use a recursive method, i.e., calculating them for a subset of a portfolio, and adding companies one by one. The calculation cost grows by N^2 for $X_n(h.)$, N^3 for $\tilde{Y}_{kn}(h.)$, and $N^3 \times M$ for $V_i^{nm}(h.)$.

We can state the following proposition for (M2).

Proposition 8 *Suppose an independent-default measure $Q_I \in \mathcal{Q}_I \cap \mathcal{Q}(Q_0, \tilde{\mathcal{F}})$ that holds (23) and $h_k = Q_I(D^k_{T_M} = 1), k = 1, \ldots, N$. $\hat{Q}$ minimizes $G(Q)$ in $(Q_I, \tilde{\mathcal{G}}_M)$-equivalent measures that hold (M2), i.e., $\mathcal{Q}(Q_I, \tilde{\mathcal{G}}_M) \cap \mathcal{Q}^2_M$ iff $\varrho.(\hat{Q}|Q_I)$ is represented by a form*

$$\varrho_n(\hat{Q}|Q_I) = \frac{\exp\Big(\sum_{l=1}^{L}\lambda_l \sum_{i=1}^{M}\sum_{m=0}^{N} A_l^{im}V_i^{nm}(h.) - \Lambda_n(h.)\Big)}{\sum_{n=0}^{N} X_n(h.)\exp\Big(\sum_{l=1}^{L}\lambda_l \sum_{i=1}^{M}\sum_{m=0}^{N} A_l^{im}V_i^{nm}(h.) - \Lambda_n(h.)\Big)} \tag{32}$$

where parameter $\lambda_l, l = 1, \ldots, L$ is determined to hold (M2'). If $\tilde{\mathcal{Q}}(Q_I, \tilde{\mathcal{G}}_M) \cap \mathcal{Q}^2_M \neq \emptyset$, there exists such $\hat{Q}$ uniquely.

From the lemma 7 and proposition 8, we can state the following corollary for the minimal entropy martingale measure among $\tilde{\mathcal{Q}}_I$.

Corollary 9 (MEMM among $\tilde{\mathcal{Q}}_I$) *If $\hat{Q}$ minimizes $G(Q)$ in $\tilde{\mathcal{Q}}_I \cap \mathcal{Q}_M$, then $\hat{Q}$ is $\tilde{\mathcal{G}}_M$-equivalent to an independent-default measure $Q_I \in \mathcal{Q}_I \cap \mathcal{Q}(Q_0, \tilde{\mathcal{F}})$ that holds (23), and $\varrho.(\hat{Q}|Q_I)$ is represented by the form (32), where $h_k = Q_I(D^k_{T_M} = 1)$ for $k = 1, \ldots, N$. If $\tilde{\mathcal{Q}}_I \cap \mathcal{Q}_M \neq \emptyset$, there exists such $\hat{Q}$.*

There exists the following redundancy in the determination of $h.$.

Remark 10 *An independent-default measure $Q'_I \in \mathcal{Q}_I \cap \mathcal{Q}(Q_0, \tilde{\mathcal{F}})$ that holds (23) is $\tilde{\mathcal{G}}_M$-equivalent to $\hat{Q}$ in proposition 8, if $h'_. = Q'_I(D^._{T_M} = 1)$ holds*

$$\ln\left(\frac{h'_k}{1-h'_k}\right) - \ln\left(\frac{h_k}{1-h_k}\right) = \ Const. \qquad for\ k = 1, \ldots, N \tag{33}$$

This remark is proved easily from

$$\begin{aligned} \rho_n(\hat{Q}|Q'_I)X_n(h'_.) &= \rho_n(\hat{Q}|Q_I)X_n(h.), & (34)\\ Y_{kn}(h'_.) &= Y_{kn}(h.) & (35)\\ V_i^{nm}(h'_.) &= V_i^{nm}(h.) & (36) \end{aligned}$$

for $n, m = 0, \ldots, N$, $k = 1, \ldots, N$, $i = 0, \ldots, M$, where $\rho.$ is given by (32). To reduce this redundancy, we require $h.$ to hold;

$$\text{(C1)} \qquad \sum_{k=1}^{N} \ln\left(\frac{h_k}{1-h_k}\right) = \sum_{k=1}^{N} \ln\left(\frac{q^k_{T_M}}{1-q^k_{T_M}}\right)$$

Therefore, the absolute value of $h.$ has no meanings for $\hat{Q}$. The absolute level of the default frequency for $\hat{Q}$ is determined by $\lambda.$, and the role of $h.$ is to determine the default probability of each name under the condition of total default. In the context of a top-down approach, the role of $h.$ is to say relating to *random thinning* in Giesecke et al. [2009].

In the rest of this section, we describe how to calibrate our model to the market. $\lambda.$ is determined to hold (M2'). To solve this non-linear equation, we apply a quasi Newton method. Meanwhile, $h_k = Q_I(D^k_{T_M} = 1), k = 1, \ldots, N$ should satisfy (M1) as

$$q^k_{T_M} = h_k \sum_{n=0}^{N} \varrho_n \tilde{Y}_{kn}(h.) \tag{37}$$

where $\rho.$ is given by (32) and

$$\tilde{Y}_{kn}(h.) := Q_I(N_{T_M} = n | D^k_{T_M} = 1) = \frac{1}{h_k} X_n(h.) Y_{kn}(h.) \tag{38}$$

Since Q_I is an independent-default measure, $\tilde{Y}_{kn}(h.)$ does not depend on h_k. Thus (37) holds by putting that

$$h_k = \frac{q^k_{T_M}}{\sum_{n=0}^{N} \varrho_n \tilde{Y}_{kn}(h.)} \tag{39}$$

where $\varrho.$ and $h.$ except h_k are fixed. In summary, we calibrate our set of parameters $\lambda.$ and $h.$ as follows.

Algorithm 11 (Calibration)

1. *Choose the initial candidate of the parameters as $h_k = q^k_{T_M}$, $\lambda_l = 0$.*

2. *Calculate $X(h.)$, $Y(h.)$ and $V(h.)$ by a recursive method.*

3. *Estimate $\lambda.$ that satisfies (M2') by a quasi Newton method.*

4. *If (M1) is satisfied, finish this procedure.*

5. *Update $h.$ based on (39).*

6. *Shift $h.$ by (33) to hold (C1).*

7. *Return to the step 2.*

4 MARKET PRACTICE

We have fitted our model to the 5-year CDX NA IG series 13 on October 6, 2009 shown in Exhibit 1. CDX NA IG is an equally weighted portfolio of 125 CDSs on investment grade north American companies. Exhibit 2 shows the market quotes for the 5-year CDS in the reference portfolio and a risk free yield curve. The market quotes for CDSs were less than 300 basis points except for 2 companies, American International Group and International Lease Finance.

We discretized the time axis by 1 month. We approximated that a running spread will be paid every quarter of a year from the evaluation date to the protection seller of CDSs and tranches, and their term to maturity is 5 years at the evaluation date. The recovery rate was set as 40 percent for all names in accordance with market convention. To be consistent with tranche premiums, we calculate the implied default probability of each company as follows;

1. Calculate implied default probability of each company from the market quote of its 5-year CDS displayed in a breakeven spread.

2. Calculate the upfront fee of each company's CDS with a regularized running spread (100 basis points) from the above implied default probability.

3. Add a constant to the upfront fees calculated above to adjust their average (-17.840 basis points) to the upfront fee of the index (-1.615 basis points) calculated from the quoted fee of the tranches.

4. Recalculate the implied default probability of each company from the adjusted upfront fees.

This inconsistency came because some of CDSs did not have enough market liquidity.

Exhibit 3 shows the performance of the calibration. All calculations were performed by Matlab scripts on a desktop PC with a 3.16 GHz Intel Core 2 Duo processor and 4GB of RAM. The estimation of error was satisfactorily small for the default probability of an individual company, the upfront fee of a single name CDS, and those of tranches. The most time consuming step in our calibration was the recursive calculation of $V_{\cdot}^{\cdot\cdot}(h.)$, that called for every reevaluation of $h.$.

Exhibit 4 shows the cumulative probability of the number of defaults on the marginal pricing measure Q_0 and implied pricing measure Q. A big difference is found from about 20 to 90 of the number of defaults. On Q_0, the probability that the number of defaults exceeds 20 was less than 0.1 percent. Meanwhile, it was about 3.82 percent on Q. Even the probability that the number of defaults exceeds 90 was 0.65 percent on Q. These results came from the relatively higher (absolutely smaller, negative) quoted upfront fees of the senior tranches than those of the theoretical values on Q_0. The theoretical tranche upfront fees calculated on the marginal pricing measure Q_0 are displayed at the bottom of exhibit 1.

EXHIBIT 1: Market quotes of 5-year CDX NA IG series 13 tranches on October 6, 2009

Tranche	0% to 3%	3% to 7%	7% to 10%	10% to 15%	15% to 30%	30% to 100%	Index
Quotes (%)	54.19	21.61	8.625	2.19	-2.41	-3.59	102.68
Base Corr. (%)	49.032	53.93	58.592	67.302	90.853	N.A.	
On Q_0 (%)	58.019	45.944	17.489	-1.602	-4.741	-4.754	

Notes: Quotes of all tranches are the percent of the principal that must be paid upfront in addition to a running spread of 100 bps per year. Base correlations are an implied dependence parameter of Gaussian copula of each base tranche. Base tranches are tranches that include all their junior tranches. For example, the base tranche in column 10%-15% covers 0%-15%. The bottom line shows the theoretical upfront fee for the marginal pricing measure Q_0. Source: Bloomberg

EXHIBIT 2: Market quotes of a breakeven spread of single name 5-year CDSs in the reference portfolio of CDX NA IG, and yield curve of the US swap market on October 6, 2009

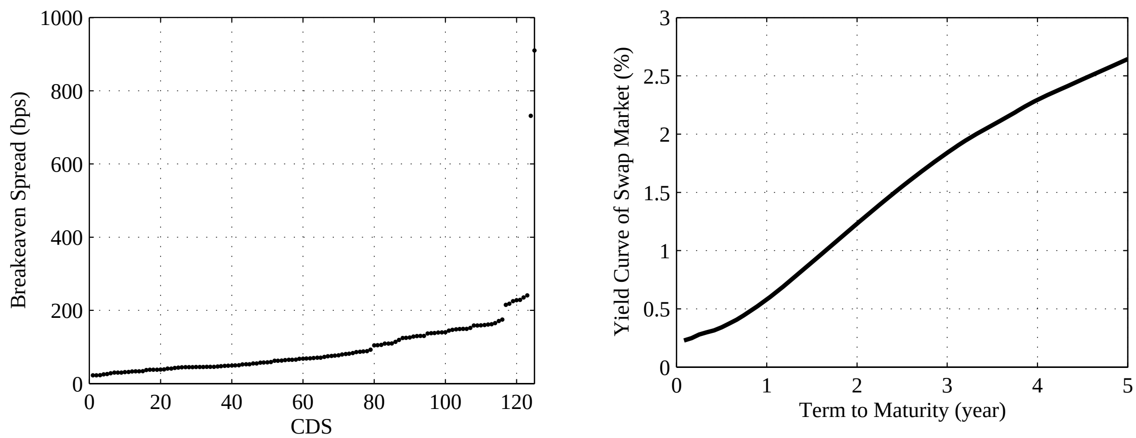
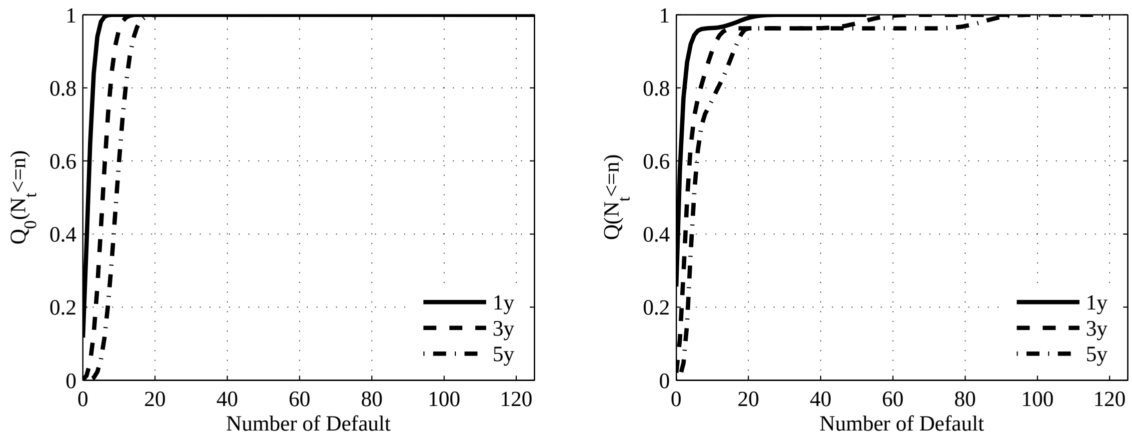


EXHIBIT 3: Performance of the calibration to 6 tranches of CDX NA IG and 125 CDSs in its reference portfolio

CPU time	8.34 sec.
Number of iterations for h.	10
Number of iterations for λ.	207
Maximum error in default prob.	$< 10^{-6}$
Maximum error in CDS up front fee	<0.004 bps
Maximum error in tranche up front fee	<0.001 bps

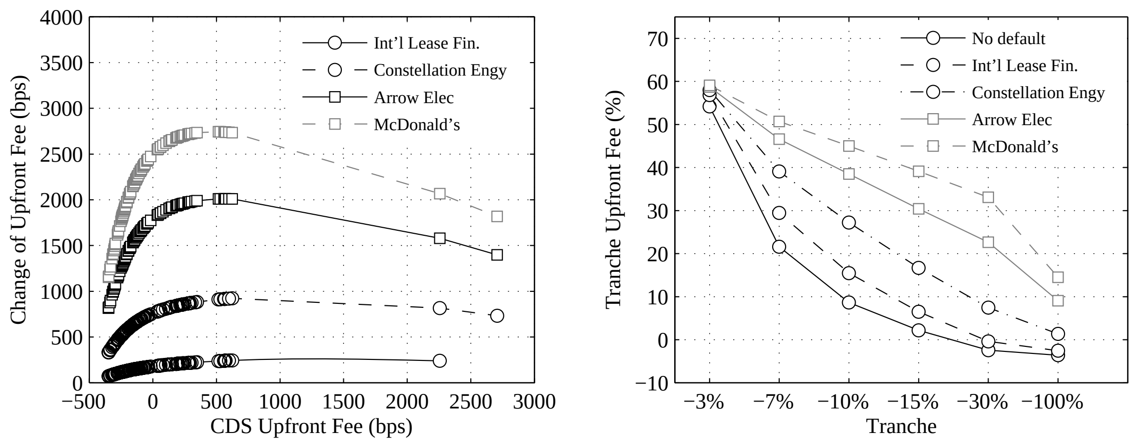
Notes: All calculations were performed by Matlab scripts on a desktop PC with a 3.16 GHz Intel Core 2 Duo processor and 4GB of RAM.

EXHIBIT 4: Cumulative probability of the number of defaults calibrated for 5-year CDX NA IG series 13 tranches on October 6, 2009



Notes: Each line shows up to 1, 3, and 5 year(s). Left: On the marginal pricing measure Q_0. Right: On the implied pricing measure Q.

EXHIBIT 5: Impact caused by a default of a company in the reference portfolio



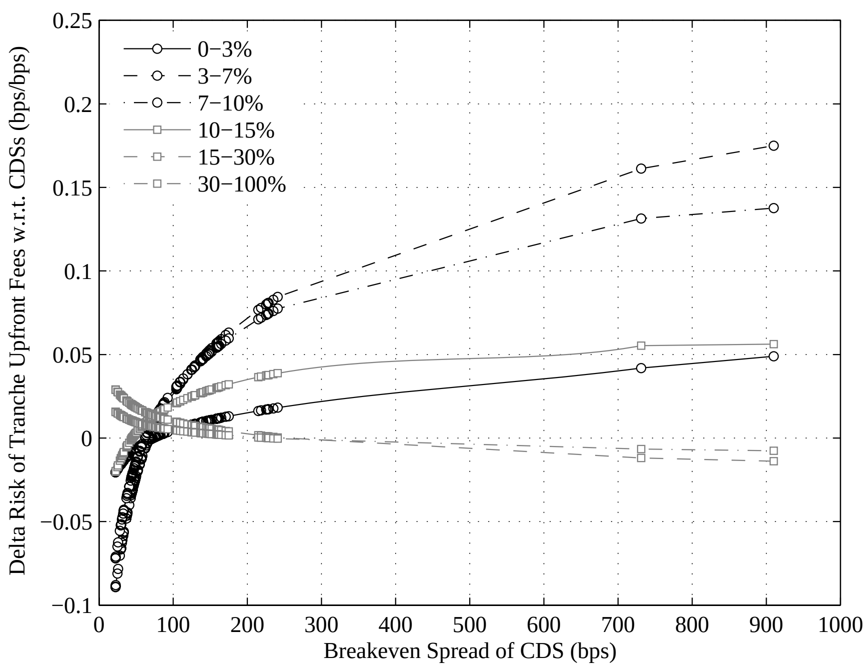
Notes: The name of the hypothetical defaulted company is given in the legend. Left: Jump of upfront the fee for a single name CDSs with a regularized running spread (100bps). Right: Up-front fee for tranches before/after a default.

EXHIBIT 6: The list of companies in hypothetical default for the default test

Company Name	Breakeven Spread	Order
McDonald's	22.36 bps	1
Arrow Elec	68.11 bps	61
Constellation Engy	228.72 bps	121
Int'l Lease Fin.	910.05 bps	125

Notes: Selected in the reference portfolio of 5-year CDX NA IG series 13, with a breakeven spread of 5 year CDS, and ranking by credit quality among the reference portfolio.

EXHIBIT 7: Delta risk on the upfront fees of credit index tranches with respect to the upfront fees of single name CDSs in the reference portfolio



We investigated *default contagion*, i.e., the impact for default probabilities caused by the default of another company. Exhibit 5 shows the jump in the upfront fees of single name CDSs with a regularized running spread (100bps) and the upfront fees of tranches before/after defaults. All parameters were estimated before the defaults. We then assumed that a company in the reference portfolio has defaulted immediately, and recalculated the upfront fees just after the default. The list of companies in hypothetical default are shown in exhibit 6. These were selected by the order of their breakeven spreads among the portfolio. A larger impact is observed from the default of *better* companies. The model suggests that even the senior tranches are exposed to risk upon the first default of *good* companies because of the strong default dependency. However, the defaults of *bad* companies cause just a small impact because their defaults are relatively not surprising.

Next, we considered *delta risk* upfront fee for tranches with respect to the upfront fees for single name CDSs. For this purpose, we have to specify the relationship between the change of the single name default probability $Q(D_t^k = 1)$ and the probability of the number of defaults $Q(N_t = n)$. In this article, we assume that the change of a single name default probability shifts a component of $\lambda_\cdot$, that corresponds to the whole credit index, and the parameter $h_\cdot$. The rest of the components of parameter $\lambda_\cdot$ are assumed to remain unchanged.

Exhibit 7 shows the ratio between the change in the upfront fee of a single name CDS with a regularized running spread (100bps) and those of tranches. We shifted the default probability of a single name company one by one, where default probabilities of the other companies were keep unchanged. The theoretical ratio of the whole credit index is $1/N = 0.008$; however, the absolute value of the ratios for tranches were significantly larger. It is worth noting that the ratios became negative in some cases, such as for *good* companies and junior tranches, or *bad* companies and senior tranches. Frey and Backhaus [2009] reported similar results. They investigated a credit portfolio constructed of two kinds (*good* or *bad*) of companies using a Markov chain model. Although this result is interesting, the *dynamics* considered here are ad hoc. We may as well take into account the change of the other components of $\lambda_\cdot$. For more on this issue, we need to specify the dynamics and investigate the time series of the market quotes.

5 CONCLUSION

In this paper we presented a new approach for modeling the default dependence among credit portfolios. We studied Minimal Entropy Martingale Measures (MEMM) for credit index tranches and derived their representation. Under a feasible assumption, we derived a tractable model for implied default dependency. Our approach has a number of advantages. First, it can calibrate all tranches belonging to a credit index and single name CDSs in its reference portfolio. Second, it has MEMM as its theoretical backbone. Third, the dynamics of default probability are excluded; as this can be an obstruction to develop dynamic versions based on our model. Fourth, our model has a minimal structure to evaluate credit index tranches, that is to say there are no latent variables. Fifth, its calculations are based on a recursive method that is quite rapid.

We have calibrated our model to market quotes of CDX NA IG tranches and showed the implied cumulative probability for the number of defaults. Using our model, we evaluated two kinds of risks on credit index tranches with respect to default contagion and spread shift.

Some implied-factor-copula models can also simultaneously calibrate tranches and CDSs and display the implied distribution of the number of defaults. The biggest difference of our approach in the model structure is that, in our context, an implied-factor-copula approach considers as many independent-default measures as the number of latent factor's states. Meanwhile, in our approach, there is only one independent-default measure determined by the minimal entropy criteria. Thus our approach has relatively little arbitrariness and can be rapidly calculated.

Our approach makes easy investigations of the structure of default dependency among credit portfolios based on the number of defaults. This realized more insightful information than does *base line correlations* of Gaussian copula. We hope that our approach is both useful in itself, and also presents a good starting point for further research into the structure of default dependence.

A Present value of credit index tranches

We show that the present values of credit index tranches become a linear function of $Q(N_{T_\cdot} = \cdot)$. First, we derive (7) as follows.

$$\begin{aligned}
p_{\text{protection}}(x) &= E^Q\Big[\sum_{i=1}^{M} L_i B_{T_i} \frac{(N_{T_i} - xN)^+ - (N_{T_{i-1}} - xN)^+}{(1-x)N}\Big] \\
&= \sum_{i=1}^{M} L_i B_{T_i} \sum_{n=0}^{N}\sum_{m=0}^{N} \frac{(n-xN)^+ - (m-xN)^+}{(1-x)N} Q(N_{T_i} = n, N_{T_{i-1}} = m) \\
&= \sum_{i=1}^{M} L_i B_{T_i} \sum_{n=1}^{N} \frac{(n-xN)^+ - (n-1-xN)^+}{(1-x)N} (Q(N_{T_{i-1}} < l) - Q(N_{T_i} < l)), \\
&= \sum_{i=1}^{M} L_i B_{T_i} \sum_{n=1}^{N} \frac{(n-xN)^+ - (n-1-xN)^+}{(1-x)N} \sum_{m=0}^{n-1} (Q(N_{T_{i-1}} = m) - Q(N_{T_i} = m)) \\
&= \sum_{i=1}^{M} L_i B_{T_i} \sum_{m=0}^{N-1} (Q(N_{T_{i-1}} = m) - Q(N_{T_i} = m)) \sum_{n=m+1}^{N} \frac{(n-xN)^+ - (n-1-xN)^+}{(1-x)N} \\
&= \sum_{i=0}^{M}\sum_{m=0}^{N-1} Q(N_{T_i} = m)\Big(1_{i<M} L_{i+1} B_{T_{i+1}} - 1_{i>0} L_i B_{T_i}\Big) \sum_{n=m+1}^{N} \frac{(n-xN)^+ - (n-1-xN)^+}{(1-x)N} \qquad (40)
\end{aligned}$$

Since if $t > s$,

$$\begin{aligned}
&\sum_{n=0}^{N}\sum_{m=0}^{N} (n-xN)^+ - (m-xN)^+) Q(N_t = n, N_s = m) \\
&= \sum_{n=0}^{N}\sum_{m=0}^{N} Q(N_t = n, N_s = m) \sum_{l=m+1}^{n} (l-xN)^+ - (l-1-xN)^+) \\
&= \sum_{l=1}^{N} ((l-xN)^+ - (l-1-xN)^+) \sum_{n=0}^{N}\sum_{m=0}^{N} 1_{m<l} 1_{n\geq l} Q(N_t = n, N_s = m) \\
&= \sum_{l=1}^{N} ((l-xN)^+ - (l-1-xN)^+) Q(N_t \geq l, N_s < l) \\
&= \sum_{l=1}^{N} ((l-xN)^+ - (l-1-xN)^+)(Q(N_s < l) - Q(N_t < l)) \qquad (41)
\end{aligned}$$

Suppose that the market information of the credit index tranche premium is given as

$$p_{\text{protection}}(x_l, y_l) - p_{\text{premium}}(x_l, y_l) = U_l, \quad (l = 1, \ldots, L) \qquad (42)$$

where x_l, y_l and U_l are the attachment point, detachment point, and upfront fee of l-th tranche, respectively. We try to present that

$$\sum_{i=0}^{M}\sum_{n=0}^{N} A_l^{in} Q(N_{T_i}=n) = p_{\text{protection}}(x_l,y_l) - p_{\text{premium}}(x_l,y_l) \tag{43}$$

and

$$b_l = U_l \tag{44}$$

The present value of legs with attachment point x and detachment point y are given by those with detachment point 1 as, respectively,

$$p_{\text{premium}}(x,y) = \frac{1-x}{y-x}p_{\text{premium}}(x) - \frac{1-y}{y-x}p_{\text{premium}}(y) \tag{45}$$

$$p_{\text{protection}}(x,y) = \frac{1-x}{y-x}p_{\text{protection}}(x) - \frac{1-y}{y-x}p_{\text{protection}}(y) \tag{46}$$

Recall that the present value of the credit index tranche premium-leg is given as

$$p_{\text{premium}}(x) = \sum_{i=1}^{M} c_i B_{T_i} \sum_{n=0}^{N}\left(1-\frac{(n-xN)^+}{(1-x)N}\right) Q(N_{T_i}=n) \tag{47}$$

Thus A_l^{in} becomes

$$A_l^{in} = \frac{1-x_l}{y_l-x_l}A^{in}(x_l) - \frac{1-y_l}{y_l-x_l}A^{in}(y_l) \tag{48}$$

where

$$\begin{aligned} A^{in}(x) &= \left(1_{i<M}L_{i+1}B_{T_{i+1}} - 1_{i>0}L_iB_{T_i}\right)\sum_{m=n}^{N-1}\frac{(m+1-xN)^+-(m-xN)^+}{(1-x)N} \\ &\quad -1_{i>0}c_iB_{T_i}\left(1-\frac{(n-xN)^+}{(1-x)N}\right) \end{aligned} \tag{49}$$

B Proof of theorem 5

Any path of $\{D_{T_i}^k; i=1,\dots,M, k=1,\dots,N\}$ can be indicated by $d\in\{0,\dots,M\}^N$ as

$$D_{T_i}^k = 1(d_k<i) \tag{50}$$

Denote the entire set of possible paths by $\mathcal{D} := \{d\in\{0,\dots,M\}^N; Q_0(d)>0\}$. For any measure $Q\in\mathcal{Q}$, denote $Q(\{D_{T_i}^k = 1(d_k<i); i,k\})$ by $Q(d)$. Then

$$Q(D_{T_i}^k = 1) = \sum_{d\in\mathcal{D}} Q(d)1(d_k<i) \tag{51}$$

$$Q(N_{T_i}=n) = \sum_{d\in\mathcal{D}} Q(d)1(\nu(d,i)=n) \tag{52}$$

where $\nu : \mathcal{D}\times\{1,\dots,M\}\to\{1,\dots,N\}$ is defined as

$$\nu(d,i) := \sum_{k=1}^{N} 1(d_k<i) \tag{53}$$

Proposition 12 *Suppose the following minimization problem for a set of non-negative values $q : \mathcal{D} \to R^+$*

$$\min_{q(\cdot)\geq 0} \tilde{G}(q), \quad \tilde{G}(q) := \sum_{d\in\mathcal{D}} q(d) \ln \frac{q(d)}{Q_0(d)} \tag{54}$$

subject to

$$\begin{aligned} \sum_{d\in\mathcal{D}} q(d) &= 1 && (55) \\ \sum_{i=1}^{M}\sum_{n=0}^{N} A_l^{in} \sum_{d\in\mathcal{D}} q(d)1(\nu(d,i)=n) &= b_l \quad (l=1,\ldots,L) && (56) \\ \sum_{d\in\mathcal{D}} q(d)1(d_k < i) &= q_i^k \quad (k=1,\ldots,N, i=1,\ldots,M) && (57) \end{aligned}$$

If there exists $q(\cdot)$ that holds conditions (55), (56) and (57), there exists a solution of this problem and it can be expressed by

$$\hat{q}(d) = \frac{Q_0(d)\exp\Big(\sum_{l=1}^{L}\sum_{i=1}^{M}\sum_{n=0}^{N}\lambda_l A_l^{in} 1(\nu(d,i)=n) + \sum_{i=1}^{M}\sum_{k=1}^{N}\zeta_i^k 1(d_k<i)\Big)}{\sum_{d\in\mathcal{D}} Q_0(d)\exp\Big(\sum_{l=1}^{L}\sum_{i=1}^{M}\sum_{n=0}^{N}\lambda_l A_l^{in} 1(\nu(d,i)=n) + \sum_{i=1}^{M}\sum_{k=1}^{N}\zeta_i^k 1(d_k<i)\Big)} \tag{58}$$

where parameters $\lambda_\cdot$ and $\zeta_\cdot^\cdot$ are determined to hold (56) and (57).

Proof. Define the following objective function with Lagrange multipliers η, $\lambda_\cdot$ and $\zeta_\cdot^\cdot$;

$$\begin{aligned} L(q) \;&:=\; \sum_{d\in\mathcal{D}} q(d)\ln\frac{q(d)}{Q_0(d)} - \eta\Big(\sum_{d\in\mathcal{D}} q(d) - 1\Big) - \sum_{l=1}^{L}\lambda_l\Big(\sum_{i=1}^{M}\sum_{n=0}^{N} A_l^{in}\sum_{d\in\mathcal{D}} q(d)1(\nu(d,i)=n) - b_l\Big) \\ &\quad - \sum_{i=1}^{M}\sum_{k=1}^{N}\zeta_i^k\Big(\sum_{d\in\mathcal{D}} q(d)1(d_k<i) - q_i^k\Big) \end{aligned} \tag{59}$$

The derivative of $L(q)$ with respect to $q(d)$ becomes

$$\frac{\partial L(q)}{\partial q(d)} = \ln\frac{q(d)}{Q_0(d)} + 1 - \eta - \sum_{l=1}^{L}\sum_{i=1}^{M}\sum_{n=0}^{N}\lambda_l A_l^{in}1(\nu(d,i)=n) - \sum_{i=1}^{M}\sum_{k=1}^{N}\zeta_i^k 1(d_k<i) \tag{60}$$

Because $L(q)$ is convex,

$$\lim_{q(d)\to 0+}\frac{\partial L(q)}{\partial q(d)} = -\infty, \quad \lim_{q(d)\to\infty}\frac{\partial L(q)}{\partial q(d)} = \infty, \quad \frac{\partial^2 L(q)}{\partial q(d)\partial q(d')} = 1_{d=d'}\frac{1}{q(d)}, \quad \frac{1}{q(d)} > 0 \tag{61}$$

there exists a unique interior solution that holds $\partial L(\hat{q})/\partial\hat{q}(d) = 0$ for all $d \in \mathcal{D}$. From condition (55), we get (58). Q.E.D.

Now let us prove theorem 5. Denote $\mathcal{Q}_D := \tilde{\mathcal{Q}}(Q_0, \tilde{F})$. For any $Q \in \mathcal{Q}_D$,

$$G(Q) = E^Q\big[\ln\rho(Q)\big] = \sum_{d\in\mathcal{D}} Q(d)\ln\frac{Q(d)}{Q_0(d)} \tag{62}$$

Since we assumed that there exists at least one pricing measure, from proposition 3, $\mathcal{Q}_D \cap \mathcal{Q}_M \neq \emptyset$. Let $Q \in \mathcal{Q}_D \cap \mathcal{Q}_M$. From (51) and (52), $q(d) = Q(d)$ holds (55), (56) and (57). Thus the problem of proposition 12 has a solution.

Using same parameters as in (17) and (58), they are related as

$$\rho(\hat{Q}) = \sum_{d\in\mathcal{D}} \frac{\hat{q}(d)}{Q_0(d)} \prod_{k=1}^{N}\prod_{i=1}^{M} 1(D^k_{T_i} = 1(d_k < i)) \tag{63}$$

So that for any $d \in \mathcal{D}$,

$$\begin{aligned}\hat{Q}(d) &= \hat{Q}(\{D^k_{T_i} = 1(d_k < i); i, k\}) \\ &= \rho(\hat{Q})|_{\{D^k_{T_i}=1(d_k<i);i,k\}} Q_0(\{D^k_{T_i} = 1(d_k < i); i, k\}) = \hat{\rho}(d)\end{aligned} \tag{64}$$

From proposition 12,

$$G(Q) = \tilde{G}(\{Q(d); d \in \mathcal{D}\}) \geq \tilde{G}(\{\hat{Q}(d); d \in \mathcal{D}\}) = G(\hat{Q}), \quad \text{for all } Q \in \mathcal{Q}_D \cap \mathcal{Q}_M \tag{65}$$

From proposition 3,

$$G(Q) \geq G(\hat{Q}), \quad \text{for all } Q \in \mathcal{Q}_M \tag{66}$$

C Proof of proposition 8

Suppose a minimization problem for $\varrho \in (R^+)^{N+1}$

$$\min_{\varrho} \hat{G}(\varrho.), \quad \hat{G}(\varrho.) := \sum_{n=0}^{N} \varrho_n X_n(h.)\big(\ln \varrho_n + \Lambda_n(h.)\big) \tag{67}$$

subject to

$$\sum_{n=0}^{N} \varrho_n X_n(h.) = 1 \tag{68}$$

$$\sum_{n=0}^{N}\sum_{m=0}^{N} \varrho_n X_n(h.) A^{im}_l V^{nm}_i(h.) = b_l, \quad (l = 1, \ldots, L) \tag{69}$$

Define the following objective function with Lagrange multipliers η, $\lambda.$;

$$L(\varrho) := \hat{G}(\varrho.) - \eta\Big(\sum_{n=0}^{N} \varrho_n X_n(h.) - 1\Big) - \sum_{l=1}^{L} \lambda_l \Big(\sum_{n=0}^{N}\sum_{m=0}^{N} \varrho_n X_n(h.) A^{im}_l V^{nm}_i(h.) - b_l\Big) \tag{70}$$

The derivative of $L(\varrho)$ with respect to ϱ_n becomes

$$\frac{\partial L(\varrho)}{\partial \varrho_n} = X_n(h.)\Big(\ln \varrho_n + \Lambda_n(h.) + 1 - \eta\Big) - X_n(h.) \sum_{l=1}^{L} \lambda_l \sum_{i=1}^{M}\sum_{m=0}^{N} A^{im}_l V^{nm}_i(h.) \tag{71}$$

Because $L(q)$ is convex,

$$\lim_{\varrho_n\to 0+} \frac{\partial L(\varrho)}{\partial \varrho_n} = -\infty, \quad \lim_{\varrho_n\to\infty} \frac{\partial L(\varrho)}{\partial \varrho_n} = \infty, \quad \frac{\partial^2 L(\varrho)}{\partial \varrho_n \partial \varrho_m} = 1_{n=m} \frac{X_n(h.)}{\varrho_n}, \quad \frac{X_n(h.)}{\varrho_n} > 0 \tag{72}$$

there exists a unique interior solution $\hat{\rho}$ that holds $\partial L(\varrho)/\partial \varrho_n = 0$ for all $n = 0, \ldots, N$. From (68), $\hat{\rho}(d)$ has the form (32).

ENDNOTES

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