Utility-Based Pricing of Defaultable Bonds and Decomposition of Credit Risk

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Abstract. We discussed utility-based pricing of defaultable bonds where their recovery values are unpredictable. By considering an optimal investment problem for bond holders, we have derived simultaneous partial integro-differential equations that the utility-based bond price solves. We have also illustrated their numerical solution. We aimed to extract credit risk premium from the yield spread of defaultable bonds and to classify them to default-timing risk, recovery risk, and spread risk.

Keywords: Default-timing risk, Recovery risk, Spread risk, Doubly-stochastic model, Backward stochastic differential equation, Marginal utility-based pricing. Utility-indifference pricing.

JEL Classifications: C65, D52, G33

1. Introduction

The risk related to default events is called *Credit risk*. Credit risk is not simple as it is a composition of risks such as default-timing risk, recovery risk, and spread risk. In this article, we are concerned with the structure of the credit risks and their premiums, especially in the context of expected-utility maximization.

Much has been written about credit risk. For example, Duffie and Singleton (2003) and Schönbucher (2003) are good texts for researchers and practitioners. Bielecki and Rutukowski (2001) described the mathematical details of credit risk models. Schönbucher (2000) and Wei (2006) survey recent work.

Let us start our discussion from the corporate bond model proposed by Merton (1974). It is the origin of the structural form approach in which bonds are the senior claims to firm values. In the Merton model, default occurs only at the maturity of the bond and the firm value is a geometric Brownian motion. Black and Cox (1976) developed the

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model so that the default time is the first hitting time of the firm value to its liability. In those models, the default time is predictable and the bond is a redundant security.

Many developments of the structural form approach derive an unpredictable default time with an unobservable default trigger[1], a noisy observation of the firm value[2], or a jump of the firm value[3]. These models have finite default intensities and so the default-timing risk is isolated from the stock price change. Therefore, the defaultable claims are no longer redundant.

These incomplete structural models are essential not distinct from the reduced form approach in which default intensity is directly considered. The models that are described by stochastic default intensity are often called types of the *doubly-stochastic model*[4]. An alternative way of modeling default intensity-dynamics is to relate it to credit-rating migration[5]. Both these models, which have different concepts, are uniformly described as conditional Markov chain models [6]. We describe defaults using a doubly-stochastic model in this article.

Another inconspicuous but important credit risk is in recovery (or Loss-Given-Default). If a defaultable claim holder cannot predict the recovery value of what they hold, they are exposed to the recovery risk. Although recovery risk is paid less attention the default-timing risk, it sometimes plays crucial role at the time of settlement of claims written on defaultable bonds or their portfolio, as with collateralized debt obligations.[7]

The simplest model of recovery is the zero-recovery model. It can also deal with bonds that recover constant value at their maturity and hence are mimicked by zero-recovery and default-free bonds. The models that express a recovery value by a predictable process may be more acceptable for practitioners. If the recovery value of a bond is in proportion to its pre-default price, it is called fractional-recovery.

Recent studies modeled unpredictable recovery values by normal distributions[8], by log-normal distributions[9], or by logit transformation

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[1] Giesecke and Goldberg (2005)
[2] Duffie and Lando (2001), Nakagawa (2000)
[3] Dao and Jeanblanc (2006), Courtois (2006)
[4] Duffie and Singleton (1999)
[5] Jarrow, Lando, and Turnbull (1997), Kijima and Komoribayashi (1998)
[6] Bielecki and Rutkowski (2003)
[7] If the expected losses are the same, low-Default-Probability and high-LGD collateral is more harmful for the senior tranche than high-DP and low-LGD ones. Hence the former is more damaging to subordinated tranches.
[8] Frye (2000)
[9] Pykhtin (2003)

of normal distributions[10]. Dllmann and Trapp (2004) compared those models empirically. Schuermann (2004) studied the historical recovery rates of individual defaults based on the market price of the defaulted bonds and showed that their distribution changes on a macroeconomic state.[11] In this article, we model recovery values using truncated normal distributions with dependence on a macroeconomic state.

Default-timing risk and recovery risk are direct risks for defaultable claim holders. There is also another kind of credit risk called *spread risk*. Spread risk is the risk of potential loss that arises from increases of information. The potential loss can be measured by present values, and the present values of defaultable bonds are usually represented by yield spreads from default free bonds. Therefore, we call it spread risk.

If there exist an infinite number of obligors, obligees are able to diversify away their credit risk and risk premiums should theoretically vanish because of the asymptotic arbitrage free principle.[12] But several empirical researches suggest that there are some credit risk premiums in real markets.

In respect to the default-timing risk premium, Driessen (2005) stated that the default intensity on the risk neutral measure is about twice that estimated compared to the physical measure in the American corporate bond market; he also included tax and liquidity adjustments. Berndt et al. (2005) reported that they found a ratio of 2.032 as the median estimate in the American default swap market.

On the other hand, Duffee (1999) concluded that the spread risk is positively priced in the corporate bond market. Farnsworth and Li (2003) did not estimate default intensity processes on the physical measure, though they found that the unconditional mean of the common part on the risk neutral measure is much higher than its observed value. Driessen (2005) also found that the common part of spread risk is positively priced, but that the firm-specific part is not. The empirical result from Feldhütter and Lando (2005), who constructed a credit rating migration model with latent factors, is that the slope factor of the spread risk is priced but the level factor is not.

The theme of this article is to investigate credit risk premiums in an incomplete market using utility-based pricing, which is the most used method of pricing in finance and economics. Bielecki and Jeanblanc (2004) discussed the utility-indifference price of defaultable claims by a backward stochastic differential equation. For other applications of utility-based pricing for credit risk, see also Collin-Dufresne and Hugonnier (2001) and Sircar and Zariphopoulou (2006).

[10] Schönbucher (2003)
[11] Hu and Perraudin (2002) and Altman et al. (2005) also pointed out this fact.
[12] Jarrow, Lando, and Yu (2005)

We consider a concrete example of the bond holders' investment problem in which the bond itself is not tradable. We derive a partial integro-differential equation that the utility-based defaultable bond prices should satisfy, and show its behavior by numerical calculations. Changing parameters, we can choose components of theoretical bond yield term structure, such that expected loss and several risk premiums. Our research may help us to decipher the market evidences of credit risk premiums shown by empirical studies.

The rest of this article is organized as follows. In section 2, we define a doubly-stochastic model of defaultable bonds with recovery risk. In section 3, we consider an optimal investment problem for defaultable bond holders. In section 4, we discuss utility-based pricing of defaultable bonds, and derive a partial integro-differential equation for it. We solve it numerically in section 5, and show the results of yield-spread decomposition. Section 6 concludes this article.

2. A Simple Model of Defaultable Bonds

In this section, we construct a simple model of defaultable bonds that are recovered unpredictably at their default time. Consider a complete probability space $(\Omega, \mathcal{F}, P)$ on which a d-dimensional standard Brownian motion $w_{\cdot} : \Omega \times [0, T] \to R^d$ up to a fixed time horizon T, and a couple of uniform random variables $(U, V) : \Omega \to [0, 1]^2$ are defined.[13] We assume that $w_{\cdot}$, U, and V are mutually independent. Define a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ as follows.[14]

$$\mathcal{F}_t := \sigma\{w_s; s \le t\}, \quad \mathcal{F} = \mathcal{F}_T \vee \sigma\{U, V\}.$$

DEFINITION 1. *Consider that there exists a stochastic vector process $X : \Omega \times [0, T] \to \mathcal{X} \subseteq R^D$ that is a strong solution of the following stochastic differential equation*[15].

$$dX_t = \hat{\mu}(X_t)dt + \hat{\sigma}(X_t)dw_t. \tag{1}$$

We regard (X_t) as a state vector.

DEFINITION 2. *Suppose a function $\hat{h} : C(\mathcal{X} \to R^{++})$ such that $E^P[\exp(-\int_0^T ds\hat{h}(X_s))] > 0$ for any $X_0 \in \mathcal{X}$. Define nondecreasing*

[13] See Protter (2003) for mathematical details.

[14] We assume that the all filtrations in this article are complete and right continuous.

[15] All hatted functions are deterministic (i.e. do not depend on $\omega \in \Omega$) in this article. Subscriptions with them denote partial derivatives, $\partial_x \hat{f}(x, y, z)$ by $\hat{f}_x(x, y, z)$ for example.

continuous processes $\Gamma, \Lambda : \Omega \times [0,T] \to R^+$*, an indicator function* $H : \Omega \times [0,T] \to \{0,1\}$*, and a stopping time* $\tau : \Omega \to [0,T] \vee \{\infty\}$ *as follows.*

$$\Gamma_t := \int_0^t \hat{h}(X_s)ds, \tag{2}$$

$$H_t := 1_{e^{-\Gamma_t} \leq U}, \tag{3}$$

$$\tau := \inf\Big\{\{t \in [0,T]; H_t = 1\} \vee \{\infty\}\Big\}, \tag{4}$$

$$\Lambda_t := \Gamma_{t \wedge \tau}. \tag{5}$$

PROPOSITION 3. *Define a stochastic process function* $M : \Omega \times [0,T] \times \mathcal{B}([0,1]) \to R$ *and a filtration* $(\mathcal{G}_t)_{t \in [0,T]}$*, which represents the information open to the market, as follows;*[16]

$$M(t,B) := 1_{V \in B} H_t - \mu(B)\Lambda_t, \tag{6}$$

$$\mathcal{G}_t := \sigma\{w_s, M(s,\cdot); s \leq t\}, \tag{7}$$

where $\mu(\cdot)$ *and* $\mathcal{B}(\cdot)$ *denote the Lesbegue measure and the Borel field, respectively, so that* $M(t,B)$ *is a* $(P,(\mathcal{G}_t))$*-martingale for any* B*.*

Proof. For the first term of equation (6), the expectation value is,

$$\begin{aligned} E[1_{V \in B} H_T | \mathcal{G}_t] &= 1_{V \in B} P(e^{-\Gamma_t} \leq U | \mathcal{G}_t) + \mu(B) P(e^{-\Gamma_T} \leq U < e^{-\Gamma_t} | \mathcal{G}_t) \\ &= 1_{V \in B} H_t + (1 - H_t)\mu(B) E[1 - e^{\Gamma_t - \Gamma_T} | \mathcal{G}_t]. \end{aligned}$$

On the other hand, for the second term, that is,

$$\begin{aligned} E[\Lambda_T | \mathcal{G}_t] &= \Lambda_t + (1 - H_t) E[E[\Lambda_T - \Lambda_t | U < e^{-\Gamma_t}] | \mathcal{G}_t] \\ &= \Lambda_t + (1 - H_t) E[1 - e^{\Gamma_t - \Gamma_T} | \mathcal{G}_t]. \end{aligned}$$

Therefore $M(t,B)$ is a $(P,(\mathcal{G}_t))$-martingale.

Q.E.D.

PROPOSITION 4. τ *is a* $(\mathcal{G}_t)$*-totally inaccessible stopping time whose intensity at time* t *is given by* $1_{\tau \geq t}\hat{h}(X_t)$*.*

Proof. For any $(\mathcal{G}_t)$-predictable stopping time S, there exists a $(\mathcal{G}_t)$-stopping time sequence $S_n < S$ such that $\lim_{n \to \infty} S_n = S$. Because S_n is $(\mathcal{G}_t)$-measurable,

$$P(S_n = s | \Gamma_s > U \geq \Gamma_v, \{\Gamma_t\}_{t \geq 0}) = P(S_n = s | \Gamma_s > U, \{\Gamma_t\}_{t \geq 0}),$$

[16] Note that $\mathcal{F} \subseteq \mathcal{G}_T$ is not guaranteed.

for any $0 \leq s < v$. The distribution of τ conditioned by S_n and $\{\Gamma_t\}_{t\geq 0}$ becomes,

$$\begin{aligned} & P(s < \tau \leq v | S_n = s, \{\Gamma_t\}_{t\geq 0}) \\ &= P(s < \tau | S_n = s, \{\Gamma_t\}_{t\geq 0}) P(\tau \leq v | s < \tau, \{\Gamma_t\}_{t\geq 0}) \\ &= P(s < \tau | S_n = s, \{\Gamma_t\}_{t\geq 0})(1 - e^{\Gamma_s - \Gamma_v}). \end{aligned}$$

Therefore the conditional probability density of τ at $v > S_n$ is finite as follows.

$$\begin{aligned} & \lim_{\delta\downarrow 0} \delta^{-1} P(v < \tau \leq v + \delta | S_n = s) \\ &= \lim_{\delta\downarrow 0} \delta^{-1} (P(s < \tau \leq v + \delta | S_n = s) - P(s < \tau \leq v | S_n = s)) \\ &= \lim_{\delta\downarrow 0} \delta^{-1} P(s < \tau | S_n = s) E[e^{\Gamma_s - \Gamma_v}(1 - e^{-\int_0^\delta du \hat{h}(X_{v+u})})] \\ &= P(s < \tau | S_n = s) E[e^{\Gamma_s - \Gamma_v} \hat{h}(X_v)] < \infty. \end{aligned}$$

Recall that $S > S_n$, we get $P(\tau = S < \infty) = 0$.

Q.E.D.

Consider that an obligor whose default time is given by τ has issued a bond that yields a unit cash flow per share at time T if the obligor has not defaulted, i.e. if $\tau > T$. If the obligor has defaulted before time T, its recovery value is to be paid for the holder per their share at the default time τ,

We define the recovery as *the fractional recovery of market value at default* as follows. We can assume that the bond price is a $(\mathcal{G}_t)$-adapted process denoted by p_t at time t, and $p_{t-} := \lim_{s\uparrow t} p_s$ is to be $(\mathcal{G}_t)$-predictable. Suppose a function $\hat{\varphi} : [0, 1] \times \mathcal{X} \to [0, 1]$ which is non-decreasing with respect to its first argument. We put the recovery value $\hat{\varphi}(V, X_\tau) p_{\tau-}$ with the random variable V. Note that the recovery value is $(\mathcal{G}_\tau)$-measurable.

3. An Optimal Investment Problem for Defaultable Bond Holders

In this section, we consider an optimal investment problem for defaultable bond holders, and derive a partial integro-differential equation that their expected utility functions solve.

Consider an investor who holds the defaultable bonds of F. We prohibit trading the defaultable bond. In other words, the investor holds the bond of constant F until its default event or maturity. Assume that the investor's terminal utility at time T is expressed by a function $\hat{U} : C^2(R^+ \to R^+)$ whose derivatives satisfy the inequities of $\hat{U}'(\cdot) > 0$ and $\hat{U}''(\cdot) < 0$.

We consider a simple market in which a set of risky assets are traded. Their prices are to be described by an n-dimensional vector process $(S_t)_{t\in[0,T]}$. Assume that they are geometric Brownian motions as;

$$dS_t^i = S_t^i(\mu_i dt + \sigma_i dw_t), \quad S_0^i = 1, \tag{8}$$

where S_t^i denotes the i^{th} component of the vector S_t, μ denotes an n-dimensional constant vector, and σ denotes an $n \times d$ constant matrix. We put $d \geq n$ and $\sigma_i \sigma_j^T = 1_{i=j}$.We assume that a bank account is also traded in the market, and its price is always 1. Thus the risk free interest rate is always 0.

Suppose the set of all admissible strategies is Π. Any $\pi \in \Pi$ is an n-dimensional vector valued $(\mathcal{G}_t)$-predictable processes that denotes a fraction of the portfolio invested in risky assets, and satisfies $E[\int_0^T \pi_t^2 dt] < \infty$. Hence the cashflow yielded by the bond up to time t is $H_t\hat{\varphi}(V, X_{\tau\wedge t})p_{(\tau\wedge t)-}$, the self-financing wealth process *excluding* the present value of the bond with any strategy $\pi \in \Pi$ solves the following stochastic differential equation with jump;

$$W_t^\pi = W_0 + \int_0^t \Big[\pi_{s-}W_{s-}^\pi(\mu ds + \sigma dw_s) + F\hat{\varphi}(V, X_s)p_{s-}dH_s\Big]. \tag{9}$$

If the issuer has not defaulted up to time T, the bond is cleared by 1. Therefore the utility maximization problem of the bond holder is expressed as follows.

$$\pi^* = \arg\sup_{\pi\in\Pi} E[\hat{U}(W_T^\pi + F(1 - H_T))]. \tag{10}$$

We assume that the bond price is given by some deterministic function $\hat{\zeta} \in C^{2,2,1}(R^+ \times \mathcal{X} \times [0,T] \to [0,1])$ as,

$$p_t = \hat{\zeta}(W_t^\pi, X_t, t). \tag{11}$$

Consider a stochastic process $Y^\pi : \Omega \times [0,T] \to R$ for each $\pi \in \Pi$ which solves the following backward stochastic differential equation[17];

$$Y_t^\pi \;=\; -\hat{U}(W_T^\pi + F(1 - H_T)) - \int_t^T \hat{\xi}(H_{s-}, W_{s-}^\pi, \pi_{s-}, X_s, s)ds$$

[17] For the general information about BSDEs in finance, see Duffie and Epstein (1992), Cvitanić and Karatzas (1993), Ma, Protter and Yong (1994), Duffie, Ma and Yong (1994), and El Karoui, Peng and Quenez (1997). Rong (1997) and Becherer (2006) studied about BSDE's with jumps. Bielecki and Jeanblanc (2004) applied BSDE to derive utility-indifference price of defaultable claims.

$$- \int_t^T \hat{z}(H_{s-}, W_{s-}^\pi, \pi_{s-}, X_s, s) dw_s$$
$$- \int_t^T \int_{[0,1]} \hat{x}(v; W_{s-}^\pi, X_s, s) M(ds, dv). \quad (12)$$

To well define Y_t^π, we assume the following.

ASSUMPTION 5. *All of* $\int_0^T ds |\hat{\xi}(H_{s-}, W_{s-}^\pi, \pi_{s-}, X_s, s)|$, $\int_0^T ds |\hat{z}(H_{s-}, W_{s-}^\pi, \pi_{s-}, X_s, s)|^2$, *and* $\int_0^T ds \int_B dv |\hat{x}(v; W_{s-}^\pi, X_{s-}, s)|$ *are finite almost surely at any* $\pi \in \Pi$, $X_0 \in \mathcal{X}$, *and* $B \in \mathcal{B}([0,1])$.

Consider functions $\hat{Y}_0 \in C^{2,2,1}(R^+ \times \mathcal{X} \times [0,T] \to R)$ and $\hat{Y}_1 \in C^{2,1}(R^+ \times [0,T] \to R)$ which satisfy terminal conditions,

$$\hat{Y}_0(W, X, T) = \hat{U}(W + F), \quad \hat{Y}_1(W, T) = \hat{U}(W),$$

and whose derivatives satisfy the following inequities at any $W \in R^+$, $X \in \mathcal{X}$, and $t \in [0,T]$;

$$\begin{array}{lll} \hat{Y}_0(W,X,t) < 0, & \hat{Y}_{0,W}(W,X,t) < 0, & \hat{Y}_{0,WW}(W,X,t) > 0, \\ \hat{Y}_1(W,t) < 0, & \hat{Y}_{1,W}(W,t) < 0, & \hat{Y}_{1,WW}(W,t) > 0. \end{array}$$

PROPOSITION 6. *The BSDE (12) has a unique solution that is expressed by* $\hat{Y}_0(\cdots)$ *and* $\hat{Y}_1(\cdots)$ *as follows,*

$$Y_t^\pi = \hat{Y}(H_t, W_t^\pi, X_t, t) = (1 - H_t)\hat{Y}_0(W_t^\pi, X_t, t) + H_t \hat{Y}_1(W_t^\pi, t), \quad (13)$$

if we put $\hat{\xi}(\cdots)$, $\hat{z}(\cdots)$, *and* $\hat{x}(\cdots)$ *as follows, respectively, and they satisfy assumption 5.*

$$\begin{aligned} \hat{\xi}(H, W, \tilde{\pi}, X, t) &:= W\tilde{\pi}\mu \hat{Y}_W(H, W, X, t) + \frac{1}{2} W^2 \tilde{\pi}^2 \hat{Y}_{WW}(H, W, X, t) \\ &\quad + \hat{\mu}(X) \hat{Y}_X(H, W, X, t) + \frac{\hat{\sigma}^2(X)}{2} \hat{Y}_{XX}(H, W, X, t) \\ &\quad + W\tilde{\pi}\sigma\hat{\sigma}^T(X) \hat{Y}_{WX}(H, W, X, t) + \hat{Y}_t(H, W, X, t) \\ &\quad + (1 - H)\hat{h}(X) \int_{[0,1]} dv \hat{x}(v; W, X, t), \quad (14) \\ \hat{z}(H, W, \tilde{\pi}, X, t) &:= W\tilde{\pi}\sigma \hat{Y}_W(H, W, X, t) + \hat{Y}_X(H, W, X, t)\hat{\sigma}(X), \quad (15) \\ \hat{x}(v; W, X, t) &:= \hat{Y}_1(W + F\hat{\varphi}(v, X)\hat{\zeta}(W, X, t), t) - \hat{Y}_0(W, X, t). (16) \end{aligned}$$

Proof. Applying the Ito formula, we can see that $\hat{Y}(\cdots)$ solves equation (12). It also satisfies the terminal condition. Hence, the coefficients do not contain Y_t^π itself and the uniqueness is trivial.

Q.E.D.

Note that if the conclusion of proposition 6 holds, Y_t^π depends on the strategy π only through W_t^π. Equation (13) means that $\hat{Y}_0(\cdots)$ and $\hat{Y}_1(\cdots)$ represent the value of Y_t^π before and after the default respectively.

Define a function $\hat{\pi} : \{0,1\} \times R^+ \times \mathcal{X} \times [0,T] \to R^n$ as follows;

$$\hat{\pi}(H,W,X,t) := -\frac{\mu}{W}\frac{\hat{Y}_W(H,W,X,t)}{\hat{Y}_{WW}(H,W,X,t)} - \frac{\sigma\hat{\sigma}^T(X)}{W}\frac{\hat{Y}_{WX}(H,W,X,t)}{\hat{Y}_{WW}(H,W,X,t)}. \tag{17}$$

THEOREM 7. *Assume that $\hat{Y}(\cdots)$ solves the following partial integro-differential equation;*

$$\begin{aligned}
&-\frac{1}{2}\frac{(\mu\hat{Y}_W(H,W,X,t) + \sigma\hat{\sigma}^T(X)\hat{Y}_{WX}(H,W,X,t))^2}{\hat{Y}_{WW}(H,W,X,t)} \\
&+\hat{\mu}(X)\hat{Y}_X(H,W,X,t) + \frac{\hat{\sigma}^2(X)}{2}\hat{Y}_{XX}(H,W,X,t) + \hat{Y}_t(H,W,X,t) \\
&+(1-H)\hat{h}(X)\int_{[0,1]} dv\hat{x}(v;W,X,t) = 0.
\end{aligned} \tag{18}$$

Moreover, assume that it satisfies the assumption 5 via equations (14), (15), and (16). If a strategy defined by a dynamic programming such that $\pi_t^{**} := \hat{\pi}(H_t, W_t^{\pi^{**}}, X_t, t)$ is admissible, then Y_t^π becomes a $(P,(\mathcal{G}_t))$-submartingale for any $\pi \in \Pi$, and only π^{**} makes $Y_t^{\pi^{**}}$ a $(P,(\mathcal{G}_t))$-martingale.

Proof. Equation (18) make $Y_t^{\pi^{**}}$ a $(P,(\mathcal{G}_t))$-local martingale hence,

$$\hat{\xi}(H,W,\tilde{\pi},X,t) = \frac{1}{2}W^2\hat{Y}_{WW}(H,W,X,t)\Big(\tilde{\pi} - \hat{\pi}(H,W,X,t)\Big)^2.$$

Moreover, assumption 5 guarantees that $Y_t^{\pi^{**}}$ becomes a $(P,(\mathcal{G}_t))$-martingale. Recall that we assumed $\hat{Y}_{WW}(\cdots) > 0$, thus the submartingale property of Y_t^π is trivial.

Q.E.D.

We give the following corollary without proof.

COROLLARY 8. *If equation (12) has a set of solutions such that Y_t^π is a $(P,(\mathcal{G}_t))$-submartingale for any $\pi \in \Pi$, and is a $(P,(\mathcal{G}_t))$-martingale* *for some unique strategy. Then, the optimal strategy and the expected utility of the investor are given as follows.*

$$\pi^* = \arg\sup_{\pi\in\Pi} E[\hat{U}(W_T^\pi + F(1-H_T))] = \arg\inf_{\pi\in\Pi} E[Y_T^\pi], \tag{19}$$

$$\sup_{\pi\in\Pi} E[\hat{U}(W_T^\pi + F(1-H_T))|\mathcal{G}_t] = -\inf_{\pi\in\Pi} E[Y_T^\pi|\mathcal{G}_t] = -Y_t^{\pi^*}. \tag{20}$$

Therefore, if we have a solution of equation (18) that gives an admissible π^{**}, then $-Y_t^{\pi^{**}}$ and π^{**} are the expected utility and the optimal strategy, respectively.

Here we note that the first term of equation (17) represents the growth optimal portfolio, and the second term represents the hedging portfolio for the increase of future default risk (i.e., the spread risk).

4. The Price of Defaultable Bonds

4.1. Utility-Based Price

We do not allow any trading of the defaultable bond as there is no market price in the usual sense. However we can consider utility-based pricing from the expected utility.

At the first step, we consider utility-indifference pricing. The utility-indifference pricing of defaultable bond is defined to keep bond holders' expected utility at latent callings (or purchases) of the bond. Therefore the utility-indifference price is denoted by a function of their wealth, state processes, and time as $\hat{q}(W, X, t)$, which satisfies,

$$\hat{Y}_1(W + F\hat{q}(W, X, t), t) = \hat{Y}_0(W, X, t), \tag{21}$$

at $t < \tau$. Hence we assumed that $\hat{Y}_1(W, t)$ is a monotonically decreasing function with respect to W and there exists an inverse function, which can be denoted by $\hat{I}(Y, t)$. $\hat{q}(W, X, t)$ is represented by $\hat{I}(Y, t)$ explicitly as follows.

$$\hat{q}(W, X, t) := \frac{1}{F}\Big(\hat{I}(\hat{Y}_0(W, X, t), t) - W\Big). \tag{22}$$

Therefore, $\hat{q}(W, X, t) \in C^{2,2,1}$.

Next, we consider marginal utility-based pricing of the defaultable bond. The basic idea of marginal utility-based pricing is well known in the fields of economics and finance. It is defined as to suppress trading for the investors. If for any agent in the market, the market price is their marginal utility-based price; as well, it is also an equilibrium-price, i.e., the market is in an equilibrium with it. Besides if we can suppose a representative investor, their marginal utility-based price is an equilibrium-price. The modern style of the definition of the marginal utility-based pricing in incomplete markets is given by Davis (1996).

In the rest of this subsection, we describe the dependence to holding amount of the bond F explicitly as $\hat{Y}_0(W, X, t; F)$ or $\hat{q}(W, X, t; F)$ and so on.

DEFINITION 9. *If a function $\hat{p}(W, X, t; F)$ satisfies the following inequality at any $F' \in R^+$ which satisfies $(F' - F)\hat{p}(W, X, t; F) \leq W$, it represents a marginal utility-based price of the defaultable bond;*

$$\hat{Y}_0(W - (F' - F)\hat{p}(W, X, t; F), X, t; F') \geq \hat{Y}_0(W, X, t; F). \qquad (23)$$

Equation (23) means that if a bond holder sells (or buys) bonds with a marginal utility-based price, their expected utility does not increase. Thus if they can trade the bond in the market at their marginal utility-based price, no trading is rational for them.

Hugonnier, Kramkov and Schachermayer (2005) argued the uniqueness of marginal utility-based prices based on the existence of equivalent martingale measures. However, as we assume it the following proposition relates the marginal utility-based bond price to the utility-indifference bond price.

PROPOSITION 10. *If there exists a marginal utility-based bond price $\hat{p}(W, X, t; F)$ uniquely, it is related to the utility-indifference bond price $\hat{q}(W, X, t; F)$ as follows.*

$$\hat{p}(W, X, t; F) = \frac{\hat{q}(W, X, t; F) + F\hat{q}_F(W, X, t; F)}{1 + F\hat{q}_W(W, X, t; F)}. \qquad (24)$$

Proof. Put $W' := W - (F' - F)\hat{p}(W, X, t; F)$. From equation (21) and $\hat{Y}_{0,W}(\cdots) < 0$, we get the following inequities,

$$\hat{p}(W, X, t; F) \begin{matrix} \geq \\ \leq \end{matrix} \frac{F'\hat{q}(W', X, t; F') - F\hat{q}(W, X, t; F)}{F' - F}, \quad \text{if} \quad F' \begin{matrix} > \\ < \end{matrix} F.$$

Therefore if the marginal utility-based bond price is uniquely well defined, it satisfies,

$$\begin{aligned} \hat{p}(W, X, t; F) &= \frac{\partial}{\partial F'}\Big(F'\hat{q}(W - (F' - F)\hat{p}(W, X, t; F), X, t; F')\Big)|_{F'=F} \\ &= \hat{q}(W, X, t; F) + F\hat{q}_F(W, X, t; F) - F\hat{q}_W(W, X, t; F)\hat{p}(W, X, t; F). \end{aligned}$$

Q.E.D.

4.2. The Case of Exponential Utility

To give a concrete calculable example, we consider that the investor's utility is expressed by an exponential form as $\hat{U}(W) = 1 - e^{-aW}$ where $a > 0$ is their risk aversion parameter. Moreover, assume that the bond price $\hat{\zeta}(\cdots)$ does not depend on their wealth. In this case, we can put,

$$\hat{Y}_0(W, X, t) = e^{-a(W+F)}\hat{f}(t)\hat{g}(X, t) - 1, \quad \hat{Y}_1(W, t) = e^{-aW}\hat{f}(t) - 1,$$

, respectively, where the terminal condition $\hat{f}(T) = \hat{g}(X, T) = 1$ holds at any $X \in \mathcal{X}$. Equation (18) leads,

$$-\frac{\mu^2}{2}\hat{f}(t) + \hat{f}_t(t) = 0, \quad \hat{f}(t) = e^{-\frac{\mu^2}{2}(T-t)},$$

and,

$$-\frac{(\sigma\hat{\sigma}^T(X))^2}{2}\frac{\hat{g}_X^2(X,t)}{\hat{g}(X,t)} + \left[\hat{\mu}(X) - \mu \cdot \sigma\hat{\sigma}^T(X)\right]\hat{g}_X(X,t)$$
$$+\frac{\hat{\sigma}^2(X)}{2}\hat{g}_{XX}(X,t) + \hat{g}_t(X,t) + \hat{h}(X)\left(e^{aF(1-\hat{r}(X,t)\hat{\zeta}(X,t))} - \hat{g}(X,t)\right) = 0, \tag{25}$$

where we put $\hat{r} : \mathcal{X} \times R^+ \to [0, 1]$ as follows to denote the *certainty equivalent* of the stochastic recovery ratio;

$$\hat{r}(X,t) := -\frac{1}{aF\hat{\zeta}(X,t)} \ln \int_0^1 dv e^{-aF\hat{\zeta}(X,t)\hat{\varphi}(v,X)}. \tag{26}$$

Note that the all assumptions about inequalities of $\hat{Y}(\cdots)$ and their derivatives with respect to W hold if $\hat{g}(\cdots) > 0$.

From equation (22), the utility-indifference price of the defaultable bond does not depend on wealth W, it is given by,

$$\hat{q}(X,t) = 1 - \frac{\ln \hat{g}(X,t)}{aF}, \tag{27}$$

or inversely,

$$\hat{g}(X,t) = e^{aF(1-\hat{q}(X,t))}. \tag{28}$$

Substituting it to equation (25), we get a partial integro-differential equation in which the utility-indifference bond price must satisfy;

$$\frac{aF}{2}\left(\hat{\sigma}^2(X) - (\sigma\hat{\sigma}^T(X))^2\right)\hat{q}_X^2(X,t) - \left[\hat{\mu}(X) - \mu \cdot \sigma\hat{\sigma}^T(X)\right]\hat{q}_X(X,t)$$
$$-\frac{\hat{\sigma}^2(X)}{2}\hat{q}_{XX}(X,t) - \hat{q}_t(X,t) + \frac{\hat{h}(X)}{aF}\left(e^{aF\hat{\zeta}(X,t)(1-\hat{r}(X,t))} - 1\right) = 0. \tag{29}$$

Hence $\hat{p}(X,t;F) = \partial_F(F\hat{q}(X,t;F))$, differentiating the equation (29) by F, we get the following partial integro-differential equation for the marginal utility-based bond price;

$$\begin{aligned}
&aF\left(\hat{\sigma}^2(X) - (\sigma\hat{\sigma}^T(X))^2\right)\hat{q}_X(X,t)\hat{p}_X(X,t) \\
&\quad - \left[\hat{\mu}(X) - \mu\cdot\sigma\hat{\sigma}^T(X)\right]\hat{p}_X(X,t) - \frac{\hat{\sigma}^2(X)}{2}\hat{p}_{XX}(X,t) - \hat{p}_t(X,t) \\
&\quad + \hat{h}(X)e^{aF\hat{\zeta}(X,t)(1-\hat{r}(X,t))}\hat{\zeta}(X,t)(1-\hat{r}(X,t)) = 0. \qquad (30)
\end{aligned}$$

Using the marginal utility-based bond price as the bond price, $\hat{\zeta}(X,t) := \hat{p}(X,t)$, Equations (29) and (30) compose simultaneous partial integro-differential equations.

5. A Numerical Example

5.1. The Default Intensity and Recovery Model

We solve equations (29) and (30) numerically, and calculate the marginal utility-based defaultable bond price. Consider that the state vector X_t is a scalar valued Ornstein-Uhlenbeck process, such that,

$$dX_t = \kappa_X(\theta_X - h_t)dt + \sigma_X dw_t, \quad \kappa_X > 0, \qquad (31)$$

and the default intensity function is given as follows.[18]

$$\hat{h}(X_t) = \exp(X_t)/10000. \qquad (32)$$

We put $\rho := \sigma\sigma_X^T/|\sigma_X|$, representing the hedgeability of the spread risk by the risky assets. If $|\rho| = 1$, the spread risk is fully hedgeable. Contrarily, if $|\rho| = 0$, there is no way to hedge the spread risk. In the case of $0 < |\rho| < 1$, the spread risk is partially hedgeable.

The bond holder is exposed to the hedgeable part of the spread risk via the risky asset portfolio, as well as the defaultable bond itself. In this article, we modeled the asset price processes of the external market by simple lognormal processes (8). However, the asset class most correlating with the defaultable bonds is the defaultable bonds themselves. So that the hedgeable part roughly corresponds to the common (systematic) part of the spread risk that represents the macroeconomic condition. On the other hand, the unhedgeable part represents the firm-specific part of the spread risk.

[18] This is called the Black-Karasinski model. Black and Karasinski (1991) applied their model to the risk free interest rate.

We put the parameters as $\sigma_X = 1.232$, $\kappa_X = 0.427$, $\theta_X = 3.219$, and $\rho = -0.482$ which are estimated by Berndt et al. (2005) for the American broadcast-entertainment firms.[19] For the external market, we put $n = 1$ and $\mu = 0.2$.

Next, let us specify the distribution of recovery values. We assume that $\hat{\varphi}(v, x)$ has a inverse function with respect to v, which is denoted by $\hat{\rho}(v, x)$. We put $\hat{\rho}(\cdots)$ as the *truncated normal distribution function* such as,

$$\begin{aligned} \int_0^v dy \hat{\rho}(y, x) &= N_{0,1}(v; \hat{b}(x), \hat{\delta}(x)) \\ &= \frac{N(v \wedge 1; \hat{b}(x), \hat{\delta}(x)) - N(0; \hat{b}(x), \hat{\delta}(x))}{N(1; \hat{b}(x), \hat{\delta}(x)) - N(0; \hat{b}(x), \hat{\delta}(x))}, \end{aligned} \tag{33}$$

where $\hat{b}(x)$ and $\hat{\delta}(x) > 0$ are some deterministic functions, and $N(v; b, \delta)$ denotes the cumulative normal distribution function with mean b and standard deviation δ.

From empirical studies about recovery[20], there exists a negative correlation between the default intensity and the recovery ratio. Table I shows the mean, standard deviation, and percentile points of the historical recovery ratio for NBER recessions and expansions.[21] We estimated parameters for the model to match means and standard deviations of the recovery ratio from historical values. Figure 1 illustrates the distribution of the recovery ratio mimicked by truncated normal distributions.

We regard that the market is in expansion if $X_t \le \theta_X$, and in recession if $X_t \ge \theta_X + \frac{\sigma_X}{\sqrt{2\kappa_x}}$, where $\frac{\sigma_X}{\sqrt{2\kappa_x}}$ is the unconditional standard deviation of X_t. If $\theta_X < X_t < \theta_X + \frac{\sigma_X}{\sqrt{2\kappa_x}}$, we simply linearly interpolate the certainty equivalent of the recovery ratio for both states. We use the above asymmetrical boundaries because the observations in periods of recession are only 332 among the total of 2035 observations.

Figure 2 shows the yield spread of the defaultable bond to the default intensity and the duration, $\hat{y}(\hat{h}(x), t) := -\frac{1}{T-t} \ln \hat{p}(x, T-t)$. All graphs illustrate that wide spreads are required for higher default intensities. The spread becomes tight as the duration becomes long at higher default intensities. Contrarily, it becomes wide at lower default intensities. This phenomenon is explained by the mean reversion prop-

[19] They estimated default intensities by Moody's KMV EDFs. In their research, they estimated the specific θ_X for each firm. We use the median of the estimation for 19 firms in the sector.

[20] Hu and Perraudin (2002), Schuermann (2004), Altman et al. (2005).

[21] This definitely denotes the distribution of the ratio between defaulted bond prices and their face values.

Table I. Recoveries across the business cycle

	Mean	Std. Dev.	Historical			Truncated normal		
			25%	50%	75%	25%	50%	75%
Recessions	32.07	26.86	10.00	25.00	48.50	11.31	26.42	49.41
Expansions	41.39	26.98	19.50	36.00	62.50	18.35	38.12	62.09
All	39.91	27.17	18.00	34.50	61.37	16.56	35.85	60.53

(Moody's, 1970-2003) from Schuermann (2004)

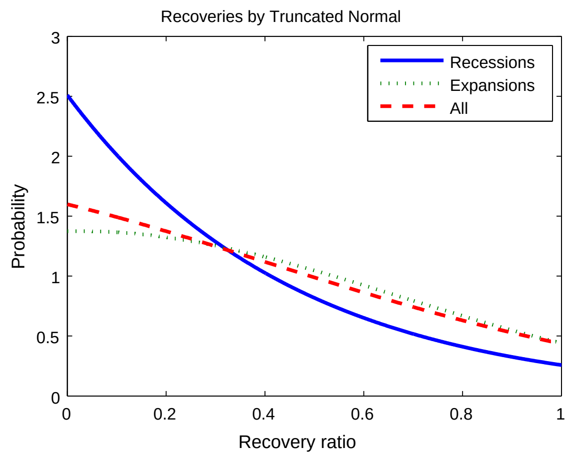

Figure 1. Recovery distribution approximated by truncated normal distribution

erty of the default intensity. Comparing the graphs, we find that the spread become wider with an increase of the bearing risk F and the risk aversion parameter a.

5.2. Spread Decomposition

The decomposition of yield spreads to risk premiums and expected loss rates suggest the risk-return structure of the defaultable bonds.

The yields of defaultable bonds contain risk free interest rates, expected loss rates, and risk premiums, which are defined the yields not involving risk free interest and expected loss rates. In our model, risk free interest rates are assumed to be zero so that the yields and the yield *spreads* are not distinguished.

Hence we have considered three kinds of credit risk. We can also categorize risk premiums for each risk. On the other hand, we can categorize risk premiums along another axis based on the relationships to the external market. The hedgeable part of the spread risk is priced *exogenously* by μ, which is the risk premium of the risky assets. The

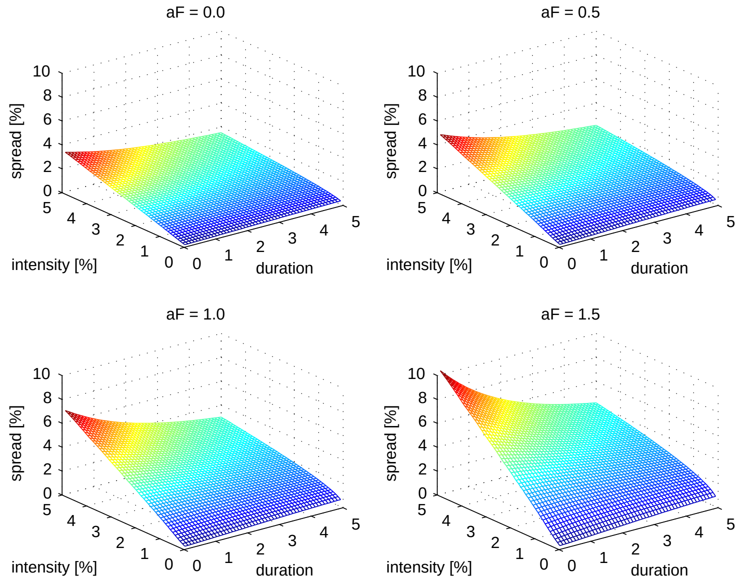

Figure 2. Yield spread of the marginal utility-based bond price [%]

unhedgeable spread risk and the other risks, the default-timing risk and recovery risk, are isolated from the external market; therefore, their prices are determined *endogenously* depending on the bond holder's utility.

Changing the parameters as in table II, we can choose the contents of the yield. Cells of *enable* use the value of table III. It is better to consider in inverse order as follows to understand this trick.

– At step 5, the yield contains all the components.

– At step 4, we change μ to zero, then the exogenous spread risk premium has vanished.

– We set $\rho = 1$ at step 3, then the unhedgeable part of the spread risk has vanished.

– At step 2, we replace the recovery value by its expectation. Thus the recovery risk premium has been removed from the yield.

– Finally, we change the bond holding amount F to zero at step 1, then all-endogenous risk premiums have vanished. Hence, the exogenous risk premium μ has already been removed and the yield is equivalent to the expected loss rate.

Table II. Contents of yield and corresponding parameters

step	contents of yield	F	recovery	ρ	μ
1	expected loss rate	0	average	1	0
2	+ default timing risk premium	enable	average	1	0
3	+ recovery risk premium	enable	enable	1	0
4	+ endogenous spread risk premium	enable	enable	empirical	0
5	+ exogenous spread risk premium	enable	enable	empirical	enable

Table III. Parameters of each figure

position	aF	recovery	μ
left-top	0.5	expansion/recession	0.2
right-top	1.0	expansion/recession	0.2
left-bottom	0.5	expansion/recession	0.5
right-bottom	0.5	all	0.2

We show the numerical results of yield-spread decomposition in figures 3 and 4, which are in expansion ($X_t = \theta_X - \frac{\sigma_X}{\sqrt{2\kappa_X}}$) and in recession ($X_t = \theta_X + \frac{\sigma_X}{\sqrt{2\kappa_X}}$), respectively. Each figure contains four graphs based on different parameters as shown at table III.

We regard the left-top (LT) as the base scenario. The right-top (RT) is the scenario where the investor is more risk averse (i.e., a is greater) or the amount of the bond F is greater. The left-bottom (LB) describes the scenario where there exists a higher exogenous spread risk premium. At the right-bottom (RB), we ignore the default intensity dependence of recovery distribution, so that those parameters for *all* in table I are used in all the range of X_t.

Figures 5 and 6 show the amounts of risk premiums relative to the expected loss rate. We also draw the conditional recovery ratio[22].

The terminal yield spread is usually regarded as the default intensity on the risk neutral measure. Berndt et al. (2005) estimated the ratio of the default intensity between physical and risk neutral measures as 1.497 (median) and 2.037 (mean). Each setting $aF = 0.5$ and $aF = 1.0$ roughly reproduced their estimation, respectively (figures 5,6 LT,RT). We also ascertained the following evidence;

[22] They are calculated by $1-(\ln p_t^b)/(\ln p_t^a)$ where each p_t^a and p_t^b denotes the bond price at step 1 with them having recovery (a) and zero recovery (b) respectively.

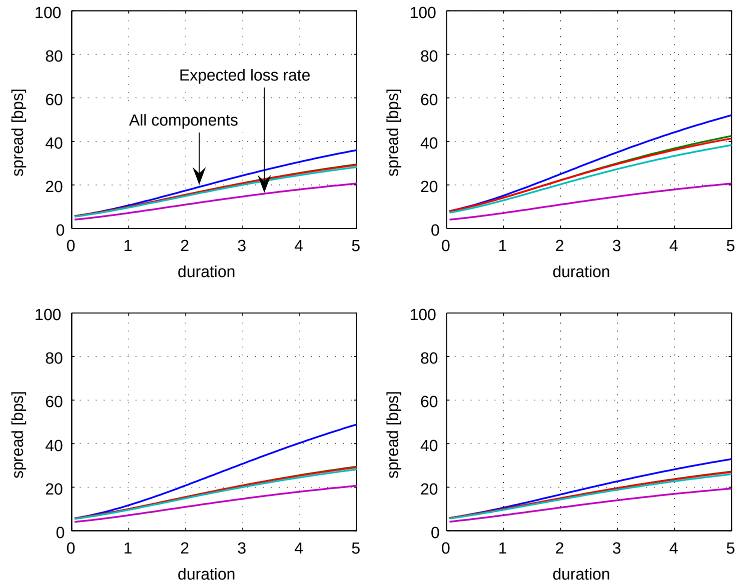

Figure 3. Decomposition of a yield spread in expansion; the stack of exogenous spread risk premium, endogenous spread risk premium, recovery risk premium, default-timing risk premium, and expected loss rate

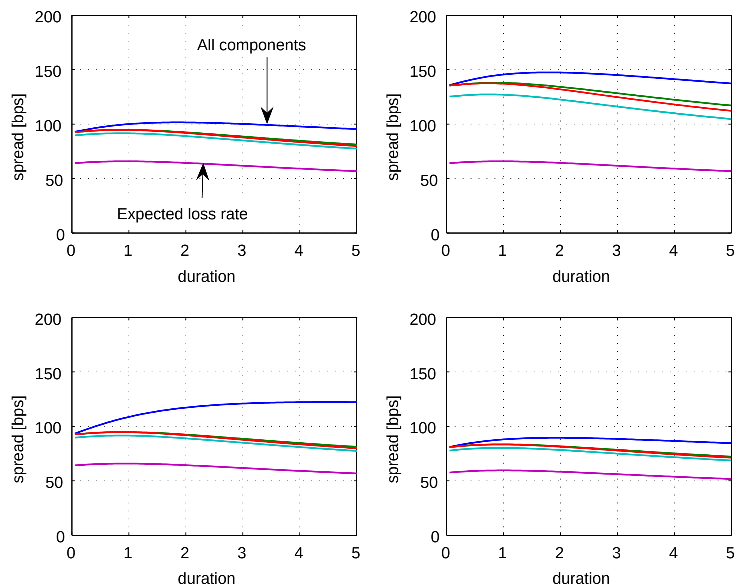

Figure 4. Decomposition of yield spread in recession (see fig. 3)

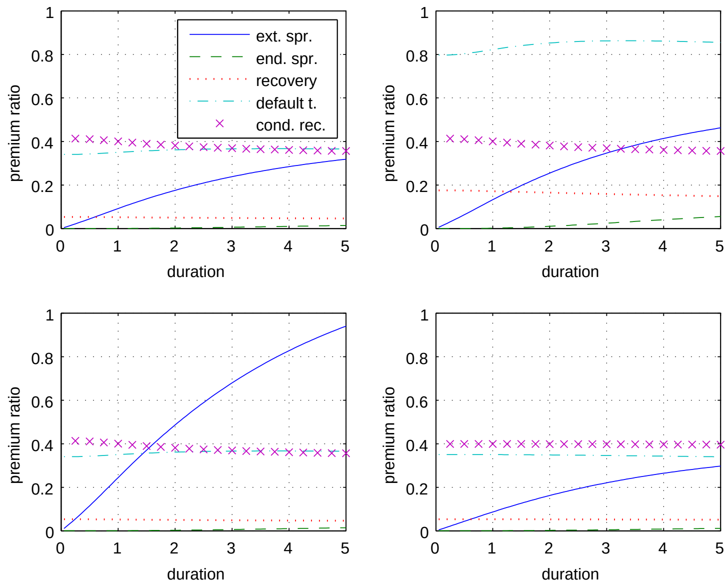

Figure 5. The value of risk premiums relative to the expected loss rate, and the conditional recovery rate in expansion (exogenous spread risk premium, endogenous spread risk premium, recovery risk premium, default-timing risk premium)

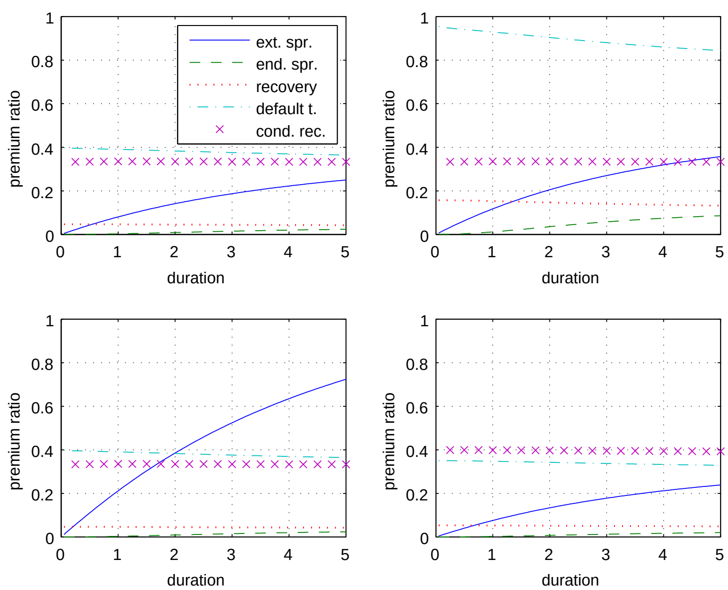

Figure 6. The value of risk premiums relative to the expected loss rate, and the conditional recovery rate in recession (see fig. 5)

1. The risk premium at a zero-duration terminal mainly consists of the default-timing risk premium.
2. The main factor of the yield spread term structure that increases is the exogenous spread risk premium.
3. The recovery and endogenous spread risk premiums are relatively small. However, they rise to become important with aF (figures 5,6 RT).
4. To reproduce an increasing term structure in a recession, a relatively high exogenous spread risk premium ($\mu \geq 0.5$) is required (figure 4 LB).
5. Correlation between the default intensity and the recovery distribution slightly reduces the conditional recovery value (figures 5,6 RB).

6. Conclusion

In this article, we have discussed utility-based pricing of defaultable bonds that was derived from an optimal investment problem of bond holders. We have then investigated the components of credit risk and their theoretical premiums.

Credit risk can be decomposed into default-timing risk, recovery risk, and spread risk. We have modeled these with two uniform random variables and Brownian motions. Each risk can be further decomposed into common and firm-specific parts. We have modeled the common part of spread risk as the hedgeable part covered by risky assets in the market. Therefore, it is priced exogenously via the processes involving risky assets. The other risks are priced endogenously by utility-based pricing.

We have derived a simultaneous partial integro-differential equation that gives a theoretical bond price, and solved it numerically with the empirically estimated parameters. Controlling the parameters, we have extracted several risk premiums from the bond yield.

Our results support the empirical findings; that the firm-specific spread risk premium is ignorable and suggest the existence of a high common spread risk premium.

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